Fusion Trees

Part One

Outline for Today

- ***Word-Level Parallelism***
  - Harnessing the intrinsic parallelism inside the processor.
- ***Word-Parallel Operations***
  - Comparing, tiling, and ranking numbers; adding and packing bits.
- ***The Sardine Tree***
  - Unconditionally beating a BST for very small integers.
- ***Most-Significant Bits***
  - Finding the most significant bit in O(1) time/space.

Working With Integers

Working with Integers

- Many practical problems involve working specifically with integer values.
  - ***CPU Scheduling:*** Each thread has some associated integer priority, and we need to maintain those priorities in sorted order.
  - ***Network Routing:*** Each computer has an associated IP address, and we need to figure out which connections are active.
  - ***ID Management:*** We need to store social security numbers, zip codes, phone numbers, credit card numbers, etc. and perform basic lookups and range searches on them.
- We've seen many general-purpose data structures for keeping things in order and looking things up.
- ***Question:*** Can we improve those data structures if we know in advance that we're working with integer data?

Working with Integers

- Integers are interesting objects to work with:
  - Their values can directly be used as indices in lookup tables.
  - They can be treated as strings of bits, so we can use techniques from string processing.
  - They fit into machine words, so we can process the bits in parallel with individual word operations.
- The data structures we'll explore over the next two lectures will give you a sense of what sorts of techniques are possible with integer data.

Our Machine Model

- We will assume we're working on a machine where memory is segmented into ***w***-bit words.
- We'll assume that the C integer operators work in constant time, and will not assume we have access to operators beyond them.

`+   -   *   /   %   <<   >>   &   |   ^   =   <=`

- Why these operations? Because they're standard across most machines. There's a bunch of papers exploring what a "reasonable" set of operations should look like, but we won't explore them here.

Some Runtime Analyses

- What are the big-O runtimes of these two pieces of code?

```
int squigglebah(unsigned int value) {
    int result = 0;
    for (int i = 0; i < sizeof(unsigned int) * 8; i++) {
        result += value & 1;
        value >>= 1;
    }
    return result;
}

void humblegwah(vector<unsigned int>& v) {
    while (true) {
        for (unsigned int& i: v) {
            if (i == 0) return;
            else i >>= 1;
        }
    }
}
```

Answer at

https://pollev.com/cs166spr23

A Key Technique: ***Word-Level Parallelism***

Word-Level Parallelism

- On a standard computer, arithmetic and logical operations on a machine word take time O(1).
- We can perform certain classes of operations (addition, shifts, etc.) on $\Theta(w)$ bits in time O(1).
  - Think of this as a weak form of parallel computation, where we can work over multiple bits in parallel with a limited set of operations.
- With some creativity, we can harness these primitives to build operations that run in time O(1) but work on $\omega(1)$ objects.
- Let’s see a quick example…

Word-Level Parallelism

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1101110	0101110	1111000	1001101	0101111	0001101	1110111	1100001

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
0011010	1000101	0010100	0100000	1010000	0100010	1000100	0001000

Word-Level Parallelism

```
  01101110 00101110 01111000 01001101 00101111 00001101 01110111 01100001
+ 00011010 01000101 00010100 00100000 01010000 00100010 01000100 00001000
-------------------------------------------------------------------------
  10001000 01110011 10001100 01101101 01111111 00101111 10111011 01101001
```

We've performed eight logical additions with a single add instruction!

The Landscape

- Preprocessing/runtime tradeoffs:

  "Yes, we have to do a lot of work, but it's a one-time cost and everything is cheaper after that."

- Randomization:

  "We might have to do a lot of work, but it's unlikely that we'll do so."

- Amortization:

  "Yes, we have to do a lot of work every once and a while, but only after a period of doing very little."

- Word-level parallelism:

  "We have to do a lot of work, but we don't have to perform many operations to do it."

Sardine Trees

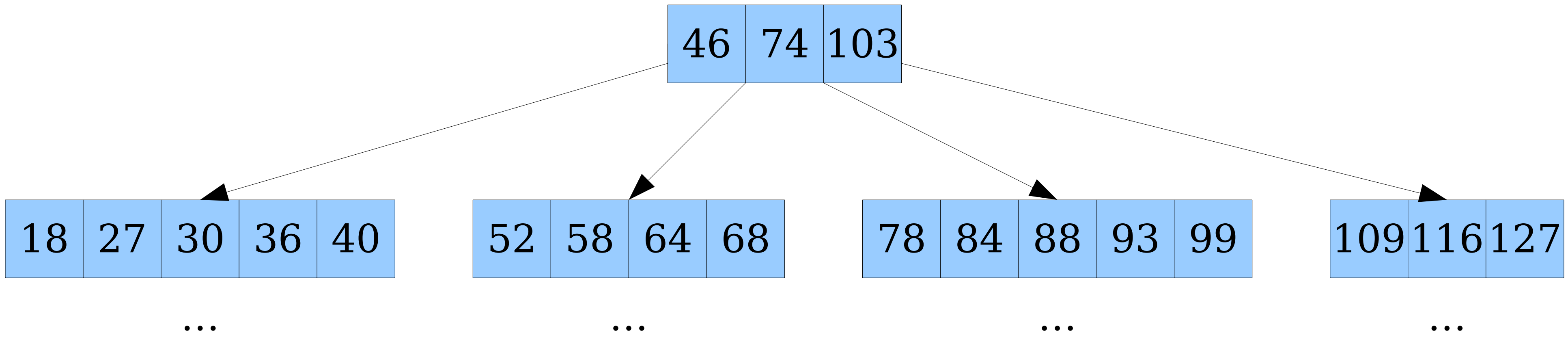
These actually aren't called sardine trees. I couldn't find a name for them anywhere and thought that this title was appropriate. Let me know if there's a more proper name to associate with them!

The Setup

- Let ***w*** denote the machine word size.
- Imagine you want to store a collection of ***s***-bit integers, where s is small compared to w.
  - For example, storing 7- bit integers on a 64-bit machine would have $s = 7$ and $w = 64$.
- Can we build an ordered dictionary that takes advantage of the small key size?

A Refresher: B-Trees

- A ***B-tree*** is a multiway tree with a tunable parameter b called the ***order*** of the tree.
- Each nodes stores $\Theta(b)$ keys. The height of the tree is $\Theta(\log_b n)$.
- Most operations (***lookup***, ***insert***, ***delete***, ***successor***, ***predecessor***, etc.) perform a top-down search of the tree, doing some amount of work per node.
- Runtime of each operation is $O(f(b) \log_b n)$, where $f(b)$ is the amount of work done per node.

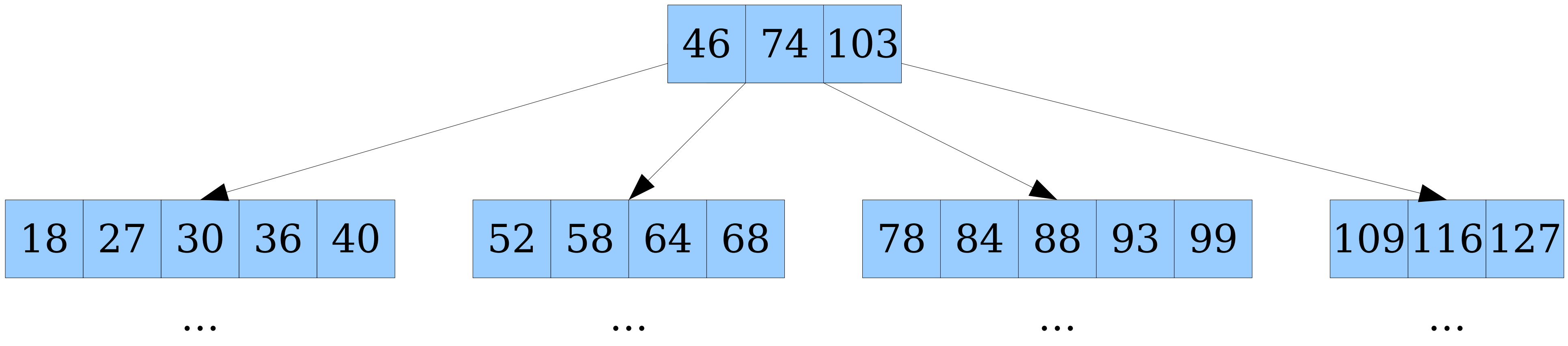


B-Tree Traversals

- Most B-tree operations work by choosing some subtree to descend into, then descending there.
- ***Claim:*** The subtree we want is given by the number of keys in the current node less than or equal to the query key k. This quantity is the ***rank*** of k.
- For example, in the top node of the B-tree shown below:

  rank(40) = 0 ***rank***(74) = 2 ***rank***(107) = 3

- ***Question:*** How quickly can we determine the rank of a key in a B-tree node?

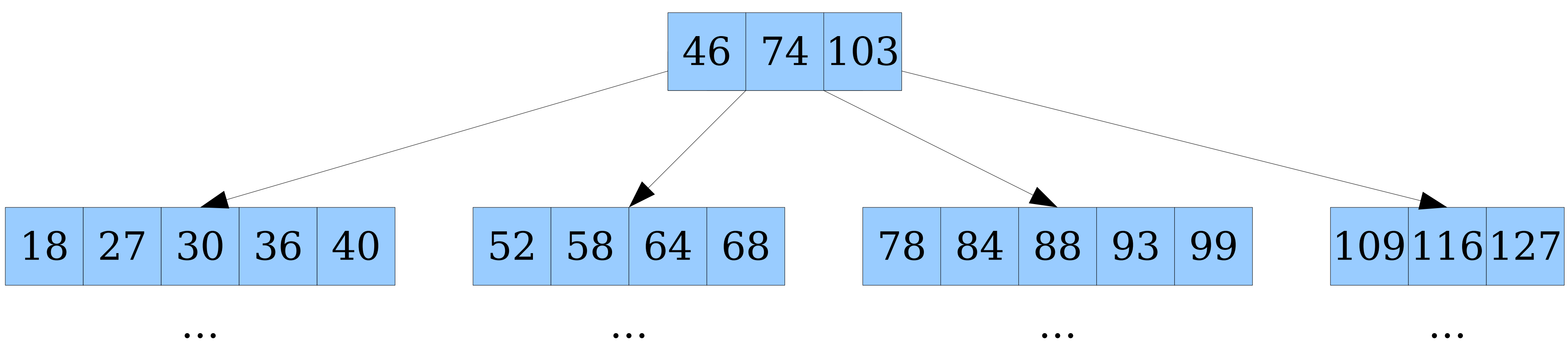


B-Tree Traversals

- We can determine ***rank***(k) with a linear search in each B-tree node for a total lookup cost of $\mathbf{O(b \cdot \log_b n)}$.
- We can determine ***rank***(k) with a binary search in each B-tree node for a total lookup cost of

$$O(\log_b n \cdot \log b) = \mathbf{O(\log n)}.$$

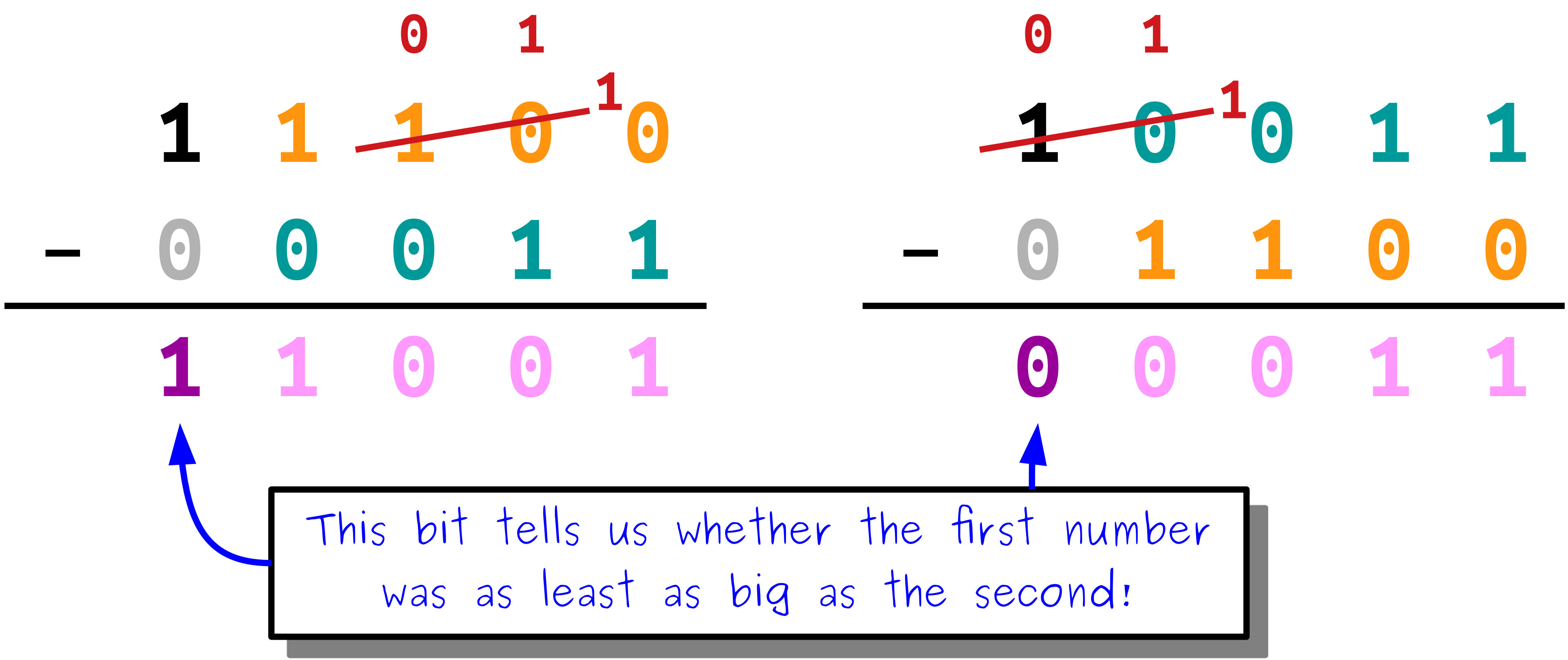
- ***Claim:*** If we can fit all the keys in a node into O(1) machine words, we can determine ***rank***(k) in time **O(1)** for total lookup cost of $\mathbf{O(\log_b n)}$.



How is this possible?

Warmup: Comparing Two Values

- Imagine we have two s-bit integers x and y and want to determine whether $x \geq y$.
- How might we do this?



Comparing Multiple Values

- This technique can be extended to work on multiple values in parallel.
- For example, here's how we'd compare eight pairs of 7-bit numbers by doing a single 64-bit subtraction:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1101110	0101110	1111000	1001101	0101111	0001101	1110111	1100001

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
0011010	1000101	0010100	0100000	1010000	0100010	1000100	0001000

Comparing Multiple Values

- This technique can be extended to work on multiple values in parallel.
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```
  11101110 10101110 11111000 11001101 10101111 10001101 11110111 11100001
- 00011010 01000101 00010100 00100000 01010000 00100010 01000100 00001000
-------------------------------------------------------------------------
  11010100 01101001 11100100 10101101 01011111 01101011 10110011 11011001
```

This technique is used in practice, including ***the glibc version of strlen***. Thanks to former CS166 student Jane Lange for pointing this out!

Fundamental Primitive: ***Parallel Compare***

Input: Two machine words. The first holds an array $x_1, \ldots x_n$ with one bit of space between each number. The second holds an array $y_1, \ldots, y_n$ with one bit of space between each number.

Output: A machine word with the result of $x_i \geq y_i$ encoded as a bit in the blank spaces between the numbers in the input array.

Procedure:

1. Use a bitwise OR to place 1s between the x_i's.
2. Use a bitwise AND to place 0s between the y_i's.
3. Compute $X - Y$. The bit preceding $x_i - y_i$ is 1 if $x_i \geq y_i$ and 0 otherwise.

Back to B-Trees

- ***Recall:*** The whole reason we're interested in making these comparisons is so that we can find how many keys in a B-tree node are less than or equal to a query key k.
- ***Idea:*** Store the (s-bit) keys in the B-tree node in a single (w-bit) machine word, with zeros interspersed:

y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
00101001	01011101	01100111	01101010	01101011	01101101	01101110	01111111

Rank in O(1)

- To perform a lookup for the key *k*, form a number by replicating *k* multiple times with 1s interspersed.

k

`1100111`

y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
00101001	01011101	01100111	01101010	01101011	01101101	01101110	01111111

Rank in O(1)

How do we do this?

- To perform a lookup for the key *k*, form a number by replicating *k* multiple times with 1s interspersed.
- Subtract the B-tree key number from it to do a parallel comparison.
- Count up how many of the sentinel bits in the resulting number are equal to 1. This is the number of keys in the node less than or equal to *k*.

Or this?

```
  11100111 11100111 11100111 11100111 11100111 11100111 11100111 11100111
- 00101001 01011101 01100111 01101010 01101011 01101101 01101110 01111111
  10111110 10001010 10000000 01111101 01111100 01111010 01111001 01101000
```

Rank: **3**

Back in Base Ten

- Suppose you have a one-digit number m.
- You want to form this base-10 number:

$$mmm$$

- Is there a nice series of arithmetical operations that will produce this?
- ***Answer:*** Compute $m \times 111$.
- Why does this work?

$$
\begin{aligned}
m \times 111 &= m \lll 2 + m \lll 1 + m \lll 0 \\
&= m00 + 0m0 + 00m \\
&= mmm.
\end{aligned}
$$

Back in Base Ten

- Suppose you have a two-digit number ***mn***.
- You want to form this base-10 number:

mnmnmn

- Is there a nice series of arithmetical operations that will produce this?
- ***Answer:*** Compute ***mn*** × 10,101.
- Why does this work?

$$
\begin{aligned}
mn \times 10{,}101 &= mn \ll 4 + mn \ll 2 + mn \ll 0 \\
&= mn0000 + 00mn00 + 0000mn \\
&= mnmnmn.
\end{aligned}
$$

Back in Base Ten

- Suppose you have a three-digit number ***mnp***.
- You want to form this base-10 number:

$$mnp000mnp0mnp$$

- Is there a nice series of arithmetical operations that will produce this?
- ***Answer:*** Compute ***mnp*** × 10,000,010,001.

$$\begin{aligned} & mnp000mnp0mnp \\ &= mnp \ll 10 + mnp \ll 4 + mnp \ll 0 \\ &= mnp \times 10^{10} + mnp \times 10^{4} + mnp \times 10^{0} \\ &= mnp \times 10{,}000{,}010{,}001 \end{aligned}$$

Fundamental Primitive: ***Parallel Tile***

Input: A number k much smaller than a machine word.

Output: A machine word holding multiple tiled copies of k, spread out with gaps between each copy.

Procedure:

1. Form a number M with a 1 bit at the end of each location to tile k.
2. Compute $M \times k$.

Computing Rank in O(1)

```
const uint64_t kMultiplier = 0b1000000010000000...100000001;
const uint64_t kOnesMask   = 0b1000000010000000...010000000;

uint64_t tiledK     = (k * kMultiplier) | kOnesMask;
uint64_t comparison = (tiledK - packedKeys) & kOnesMask;
```

```
  11100111 11100111 11100111 11100111 11100111 11100111 11100111 11100111
- 00101001 01011101 01100111 01101010 01101011 01101101 01101110 01111111
  10000000 10000000 10000000 00000000 00000000 00000000 00000000 00000000
```

How do we count how many of these bits are set?

Summing Up Flags

- After performing our subtraction, we're left with a number like this one, where the highlighted bits are "interesting" to us.
- ***Goal:*** Add up these "interesting" values using at most O(1) total operations on words.

a0000000 **b**0000000 **c**0000000 **d**0000000

An Initial Idea

- To sum up the flags, we could extract each bit individually and add the result.
- ***The catch:*** This takes time $\Theta(r)$, where r is the number of times we tiled our value.
- Can we do better?

a0000000 **b**0000000 **c**0000000 **d**0000000

A Shifty Solution

- Given this number:

  `a0000000 b0000000 c0000000 d0000000`

  we want to compute `a` + `b` + `c` + `d`.
- We can't efficiently isolate `a`, `b`, `c`, and `d`.
- ***Claim:*** We don't have to!

```
  a0000000 b0000000 c0000000 d000000
           a0000000 b0000000 c000000
                    a0000000 b000000
+                            a0000000 b0000000 c0000000 d0000000
  ???????? ???????? ??????sum??????? ???????? ???????? ????????
```

This is a series of shifts and adds. It's equivalent to multiplying our original number by some well-chosen spreader!

Fundamental Primitive: ***Parallel Add***

Input: A machine word with “interesting” bits spaced evenly across the word.

Output: The sum of those “interesting” bits.

Procedure:

1. Perform a ***parallel tile*** with an appropriate multiplier to place all leading bits on top of one another.
2. Use a bitmask and bitshift to isolate those bits.

Computing Rank in O(1)

```
const uint64_t kMultiplier = 0b1000000010000000...100000001;
const uint64_t kOnesMask   = 0b1000000010000000...010000000;

uint64_t tiledK     = (k * kMultiplier) | kOnesMask;
uint64_t comparison = (tiledK - packedKeys) & kOnesMask;

const uint64_t kStacker = 0b1000001000001...1000001;
const uint8_t  kShift   = 31;
const uint64_t kMask    = 0b111;

uint64_t rank = ((comparison * kStacker) >> kShift) & kMask;
```

```
      a0000000 b0000000 c0000000 d0000000 00000000 00000000 00000000
               a0000000 b0000000 c0000000 d0000000 00000000 00000000
                        a0000000 b0000000 c0000000 d0000000 00000000
+                                a0000000 b0000000 c0000000 d0000000
--------------------------------------------------------------------
                                                                 sum
```

Fundamental Primitive: ***Parallel Rank***

Input: An array of integers packed into a machine word with one bit of space between integers, and a key k.

Output: How many elements of the array are less than or equal to k.

Procedure:

1. Perform a ***parallel tile*** to create n copies of the key k, prefixed by 1's.
2. Perform a ***parallel compare*** of the key k against values $x_1, \ldots, x_n$.
3. Perform a ***parallel add*** to sum those values into some total t.
4. Return t.

The Sardine Tree

- Let w be the word size and s be some (much) smaller number of bits.
- A ***sardine tree*** is a B-tree of order $\Theta({}^{w}/_{s})$ where the keys in a node are packed into a single machine word.
  - Get it? The keys are "packed" tightly into a machine word! I'm funny.
- Each node is annotated with several values (the masks and multipliers from the preceding slide), which are updated in time O(1) whenever a key is added or removed.
- Supports all ordered dictionary operations in time

$$O(\log_b n) = \mathbf{O(\log_{w/s} n)}.$$

The Scorecard

- Here's the performance breakdown for the sardine tree.
- Notice that the runtime performance is *strictly better* than that of a BST!
- Notice that the space usage, as measured in words, is *sublinear*, since each node stores multiple keys!

The Sardine Tree

- ***lookup***: $O(\log_{w/s} n)$
- ***insert***: $O(\log_{w/s} n)$
- ***delete***: $O(\log_{w/s} n)$
- ***max***: $O(\log_{w/s} n)$
- ***succ***: $O(\log_{w/s} n)$
- Space: $\Theta(n \cdot {}^{s}/_{w})$ words

What's Next

- ***Question:*** Can we get performance along these lines even if the keys fill full machine words?
- The strategy used in the sardine tree on its own won't get us there – but many of those same techniques will!
- We'll see how to do this next time. In the meantime, let's see some other cool tricks we can do with word-level parallelism.

Mystery Structure?

- ***lookup***: $O(\log_w n)$
- ***insert***: $O(\log_w n)$
- ***delete***: $O(\log_w n)$
- ***max***: $O(\log_w n)$
- ***succ***: $O(\log_w n)$
- Space: $\Theta(n)$

Word-Level Parallelism Tricks #2:
Most-Significant Bits

Most-Significant Bits

- The ***most-significant bit*** function, denoted **`msb`(n)**, outputs the index of the highest 1 bit set in the binary representation of number n.
- Some examples:

  `msb(0110)` = 2 `msb(010100)` = 4 `msb(1111)` = 3

- Note that `msb(0)` is undefined.
- Mathematically, `msb`(n) is the largest value of k such that $2^k \leq n$. *(Do you see why?)*

Most-Significant Bits

- Although we didn't have this name earlier in the quarter, you've seen a place where we needed to efficiently compute $\texttt{msb}(n)$.
- Do you remember where?
- ***Answer:*** In the sparse table RMQ structure, where computing RMQ(i, j) requires computing the largest number k where $2^k \leq j - i + 1$.
- That's exactly the value of $\texttt{msb}(j - i + 1)$!

Most-Significant Bits

- On many architectures, there's a single assembly instruction that computes `msb`(*n*).
  - on x86, it's **`BSR`** (***b***it ***s***can ***r***everse).
- On others, nothing like this exists.
  - Older versions of MIPS, for example.
- ***Question:*** How would we compute `msb`(*n*) assuming we only have access to the regular C operators?

```
+   -   *   /   %   <<   >>   &   |   ^   ==   <=
```

Computing `msb`

- In Problem Set 1, you (probably) computed `msb`(n) by building a lookup table mapping each value of n to `msb`(n).
- ***The Good:*** This takes time O(1) to evaluate.
- ***The Bad:*** The preprocessing time, and space usage, is $\Theta(U)$, where U is the maximum value we'll be querying for.
- ***The Ugly:*** In the worst case $U = 2^w$.
- Can we do better?

Most-Significant Digits

- Can you compute most-significant digits
  - … in time O(w) using O(1) space?
  - … in time O(log w) using O(1) space?
  - … in time O(1) using O(1) space?
- Remember that the word size w is not a constant and that we can only use C-style operations.

Answer at

https://pollev.com/cs166spr23

Most-Significant Bits

- There's a simple O(*w*)-time algorithm for computing `msb`(*n*) that just checks all the bits until a 1 is found:

```
for (uint8_t bit = 64; bit > 0; bit--) {
    if (n & (uint64_t(1) << (bit - 1))) {
        return bit;
    }
}
flailAndPanic();
```

- Can we do better?

Computing `msb`

- We can improve this runtime to O(log *w*) by using a binary search:
  - Check if any bits in the upper half of the bits of *n* are set.
  - If so, recursively explore the upper half of *n*.
  - If not, recursively explore the lower half of *n*.
- We can test whether any bit in a range is set by ANDing with a mask of 1s and seeing if the result is nonzero:

```
  11011100 10111011 11000100 11010101 11100110 11110111 11000010 00110010
∧ 11111111 11111111 11111111 11111111 00000000 00000000 00000000 00000000
-------------------------------------------------------------------------
  11011100 10111011 11000100 11010101 00000000 00000000 00000000 00000000
```

- Can we do better?

Claim: For any machine word size w, there is an algorithm that uses O(1) machine operations and O(1) space – independently of w – and computes `msb`(n).

This is not obvious!

How is this possible?

Not Starting from Scratch

- We’re not going into this problem blind. We’ve seen a bunch of useful techniques so far:
  - ***Parallel compare:*** We can compare a bunch of small numbers in parallel in O(1) machine word operations.
  - ***Parallel tile:*** We can take a small number and “tile” it multiple times in O(1) machine word operations.
  - ***Parallel add:*** If we have a bunch of “flag” bits spread out evenly, we can add them all up in O(1) machine word operations.
  - ***Parallel rank:*** We can find the rank of a small number in an array of small numbers in O(1) machine word operations.
- This is an impressive array of techniques. Let’s see if we can reuse or adapt them.

MSBs as Ranks

- ***Recall:*** `msb`(n) is the largest value of k for which $2^k \leq n$.
- ***Idea:*** Imagine we have an array of all the powers of two that we can represent in a machine word. Then `msb`(n) is the rank of n in that array!

```
00000000 00000000 00100000 00000010 00000000 00100000 00100001 00101111
```

2^0	2^1	2^2	2^3	2^4	2^5	2^6	…	2^{50}	2^{51}	2^{52}	2^{53}	2^{54}	2^{55}	2^{56}	2^{57}	2^{58}	…	2^{61}	2^{62}	2^{63}

The Problem

- We can compute the rank of a value in an array assuming that all the array entries fit into a single machine word.
- This isn't the case here:
  - w total powers of two to write out.
  - Total bits needed: $\Theta(w^2)$, way too big to fit into a word.
- ***Question:*** Can we still harness the benefits of this parallel rank operation?

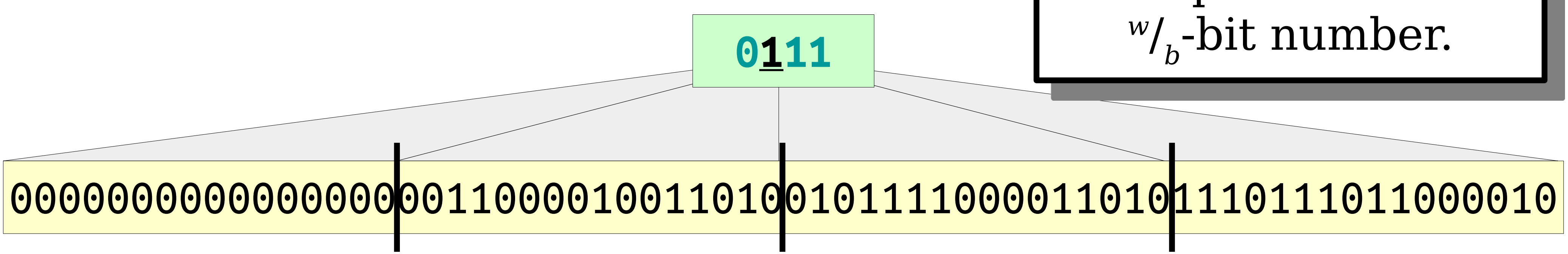
00000000 00000000 00100000 00000010 00000000 00100000 00100001 00101111

2^0	2^1	2^2	2^3	2^4	2^5	2^6	…	2^{50}	2^{51}	2^{52}	2^{53}	2^{54}	2^{55}	2^{56}	2^{57}	2^{58}	…	2^{61}	2^{62}	2^{63}

A Nice Decomposition

- Imagine we want to compute the most-significant bit of a *w*-bit integer.
  - In what follows, we'll pick $w = 64$, but this works for any w.
- We ultimately want to be finding the MSB of numbers with way fewer than *w* bits.
- ***Idea:*** Split *w* into some number of blocks of size *b*. Then,
  - find the index of the highest block with at least one 1 bit set, then
  - find the index of the highest bit within that block.

Compute `msb` for a $^{w}/_{b}$-bit number.



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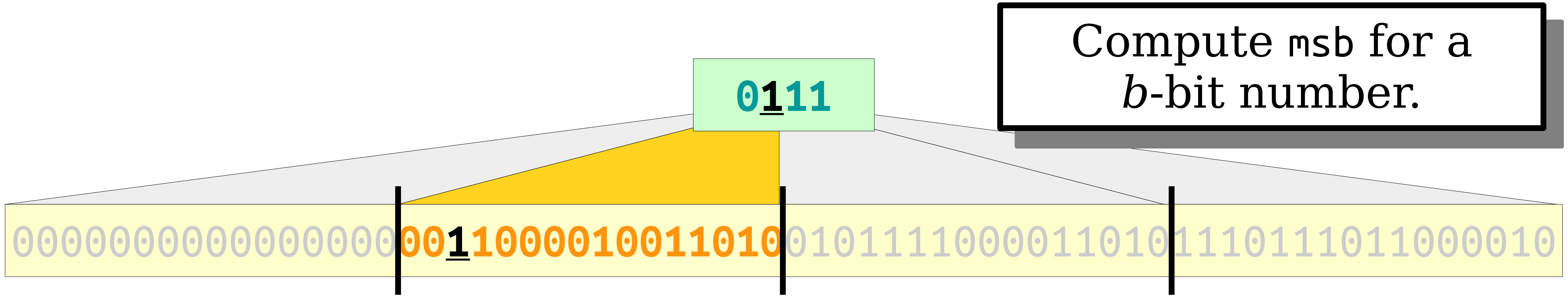
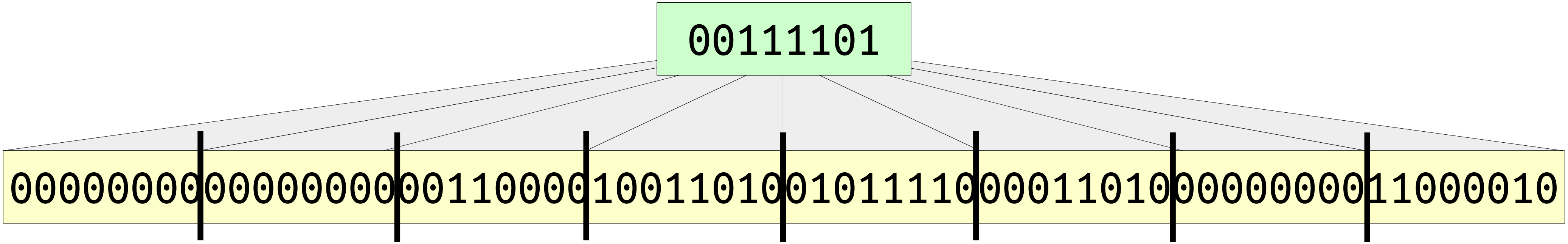


A Nice Decomposition

- We will compute the MSB for w-bit integers by solving MSB for b and $^{w}/_{b}$-bit integers.
- What choice of b minimizes $\max\{b, {}^{w}/_{b}\}$?
- ***Answer:*** Pick $\boldsymbol{b = w^{1/2}}$.
- So now we need to see how to
  - solve `msb`(n) for integers with $w^{1/2}$ bits, and
  - replace each block with a bit indicating whether that block contains a 1.



MSB for $w^{½}$ Bits

- ***Recall:*** We can compute `msb`(n) by counting how many powers of two are less than or equal to n.
- If our numbers have size $w^{½}$, there are $w^{½}$ powers of two to compare against.
- Each of those powers of two has $w^{½}$ bits, so all of those powers of two can be packed into a single machine word!
- ***Idea:*** Use our O(1)-time rank algorithm!

00000001 00000010 00000100 00001000 00010000 00100000 01000000 10000000

MSB for $w^{1/2}$ Bits

- If our numbers have size $w^{1/2}$, there are $w^{1/2}$ powers of two to compare against, each of which has $w^{1/2}$ bits.
- Our parallel comparison prepends an extra bit to each number to compare.
- That's barely – just barely – too many bits to fit into a machine word.

`000000001000000010000000100000001000000010000000100000001000000010000000`

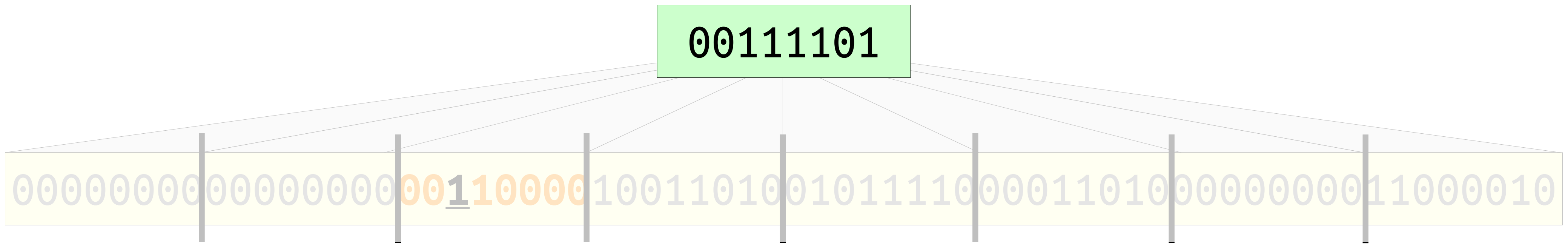
MSB for $w^{½}$ Bits

- ***Claim:*** This is an engineering problem at this point.
- ***Option 1:*** Split the powers of two into two different machine words and do two rank calculations.
- ***Option 2:*** Special-case the most-significant bit to reduce the number of bits to check.
- Either way, we find that the work done here is O(1) machine operations, with no dependency on the word size *w*!

000000001000000010000000100000001000000010000000100000001000000010000000

A Nice Decomposition

- We need to see how to
  - solve `msb`(n) for integers with $w^{½}$ bits, and
  - replace each block with a bit indicating whether that block contains a 1.



Identifying Active Blocks

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
00000000	00011001	11110000	10011010	10000000	00011010	11101110	11000010

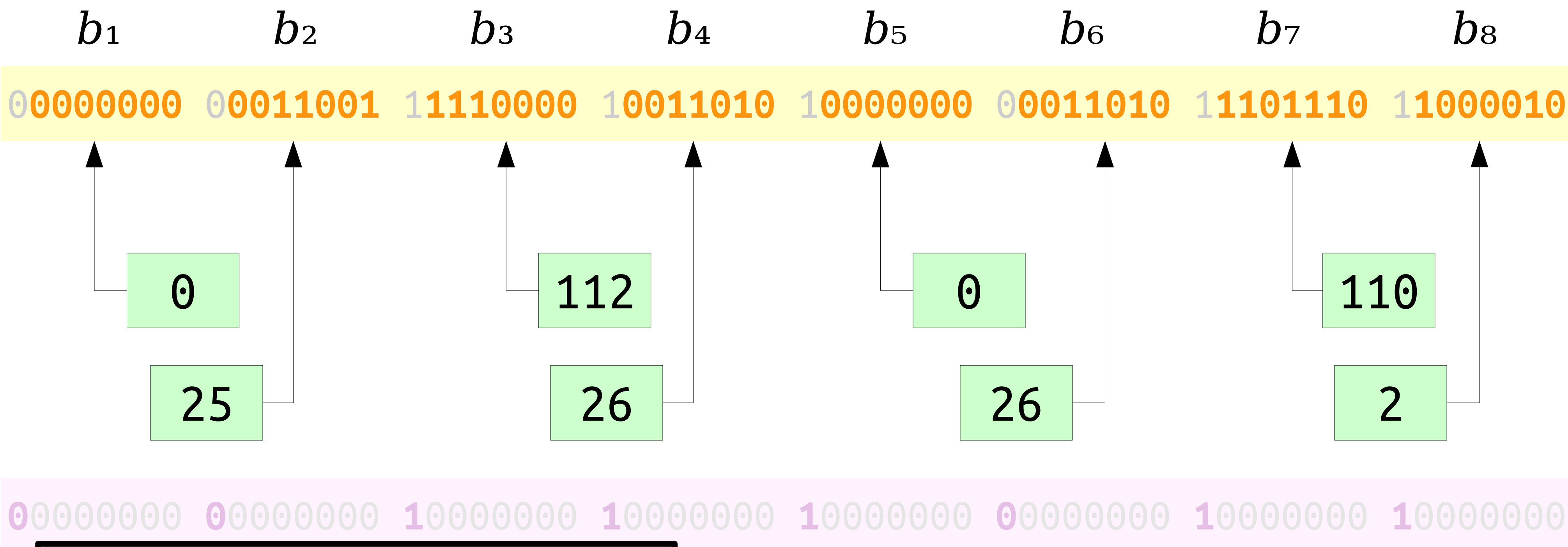
Observation: A block contains a 1 bit if its first bit is 1 or its lower 7 bits contain a 1.

Identifying Active Blocks

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
	00000000	00011001	11110000	10011010	10000000	00011010	11101110	11000010
∧	10000000	10000000	10000000	10000000	10000000	10000000	10000000	10000000
	00000000	00000000	10000000	10000000	10000000	00000000	10000000	10000000

Observation: A block contains a 1 bit if its first bit is 1 or its lower 7 bits contain a 1.

Identifying Active Blocks



A number's lower 7 bits contain a 1 if and only if the numeric value of those bits is at least 1.

Observation: A block contains a 1 bit if its first bit is 1 or its lower 7 bits contain a 1.

Identifying Active Blocks

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
	10000000	10011001	11110000	10011010	10000000	10011010	11101110	11000010
–	00000001	00000001	00000001	00000001	00000001	00000001	00000001	00000001
	01111111	10011000	11101111	10011001	01111111	10011001	11101101	11000001
	00000000	00000000	10000000	10000000	10000000	00000000	10000000	10000000

High bit set?

Observation: A block contains a 1 bit if its first bit is 1 or its lower 7 bits contain a 1.

Identifying Active Blocks

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
00000000	00011001	11110000	10011010	10000000	00011010	11101110	11000010
00000000	10000000	10000000	10000000	10000000	10000000	10000000	10000000

Any bits set?

00000000	10000000	10000000	10000000	00000000	10000000	10000000	10000000

Low bits set?

00000000	00000000	10000000	10000000	10000000	00000000	10000000	10000000

High bit set?

Observation: A block contains a 1 bit if its first bit is 1 or its lower 7 bits contain a 1.

Identifying Active Blocks

- We now have a word holding flags telling us which blocks have a 1 bit set.
- We need to find the highest set flag.
- There are only $w^{½}$ flags. If we could compact them into $w^{½}$ adjacent bits, we could use our earlier algorithm to find the highest one set!

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
00000000	00011001	11110000	10011010	10000000	00011010	11101110	11000010
00000000	**1**0000000	**1**0000000	**1**0000000	**1**0000000	**1**0000000	**1**0000000	**1**0000000

Identifying Active Blocks

- ***Idea:*** Adapt the shifting technique we used to compute ranks.
- Instead of shifting the bits on top of one another, shift the bits next to one another:

```
a0000000b0000000c0000000d0000000

  a0000000b0000000c0000000d0000000000000000000000000000
         a0000000b0000000c0000000d0000000000000000000000
                a0000000b0000000c0000000d000000000000000
+                      a0000000b0000000c0000000d0000000
  -----------------------------------------------------
  ?????????????????????abcd????????????????????????????
```

Fundamental Primitive: ***Parallel Pack***

Input: A machine word containing several "interesting" bits that are evenly spaced apart.

Output: A machine word with those "interesting" bits placed adjacent to one another at the low end of the word.

Procedure:

1. Perform a ***parallel tile*** with an appropriate multiplier to place all leading bits adjacent to one another.
2. Use a bitmask and bitshift to isolate those bits.

Putting It All Together

- Use a bitmask to identify all blocks whose high bit is set.
- Use a ***parallel tile*** and a ***parallel compare*** to identify all blocks with a 1 bit aside from the first.
- Use a ***parallel pack*** to pack those bits together.
- Use a ***parallel rank*** to determine the highest of those bits set, which gives the block index.
- Use a ***parallel rank*** to determine the highest bit set within that block.

The Finished Product

- I’ve posted a link to a working implementation of this algorithm for 64-bit integers on the course website.
- Feel free to check it out – it’s really magical seeing all the techniques come together!

What We Covered

- We can use bit-parallel tricks to
  - compare multiple values in parallel,
  - tile a number across a word,
  - sum up evenly-spaced bits in a word,
  - compute ranks in an array,
  - compact evenly-spaced bits in a word, and

  all in O(1) machine word operations!
- Using these techniques, we can modify a B-tree to work strictly faster than a conventional BST, provided that we store tiny keys.
- Using these techniques, can we compute the most-significant bit of a machine word in time O(1), independent of the machine word size.
- And all of this flows from one source: ***word-level parallelism*** inside of the processor!

What's Next

- Can we build a data structure for integers that is *strictly better* than a binary search tree?
- The answer is ***yes***, and it's called a ***fusion tree***.
- Today's exploration provides the techniques we'll use to build the fusion tree. We just need a few more insights to get us there!

Next Time

- ***Patricia Codes***
  - Compressing a small number of big integers into a small number of small integers.
- ***Fusion Trees***
  - Combining all these techniques together!