Disjoint-Set Forests

Outline for Today

- ***Iterated Functions***
  - Making an implicit idea explicit.
- ***Incremental Connectivity***
  - Finding connected nodes as a graph changes.
- ***Disjoint-Set Forests***
  - A surprisingly simple and subtle data structure.
- ***Analyzing Disjoint-Set Forests***
  - A clever, nuanced analysis with a surprising result.

Iterated Functions

Iterated Functions

- Recursive functions work by converting a problem of size *n* into one or more subproblems of a smaller size.
- How much smaller those subproblems are indicates how many levels of recursion we'll have.
  - $n \to n - 1$: $\Theta(n)$ levels.
  - $n \to n/2$: $\Theta(\log n)$ levels

Iterated Functions

- Let f be a function. The ***iterated function of f***, denoted ***$f^\star$*** is a function defined as follows:

$$f^*(n) = \begin{cases} 0 & \text{if } f(n) \leq 1 \\ 1 + f^*(f(n)) & \text{otherwise} \end{cases}$$

- Intuitively, $f^*(n)$ is (roughly) the number of times you need to apply f to n to reduce it to a sufficiently small constant.
  - If $f(n) \leq 1$, no steps are needed.
  - Otherwise, you need one step to turn n into $f(n)$, then $f^\star(f(n))$ more steps from there.

Iterated Functions

	$f^*(n)$	As seen in…
$f(n) = n - 1$	$\Theta(n)$	Linear search
$f(n) = {}^{n}/_{2}$	$\Theta(\log n)$	Binary search
$f(n) = n^{1/2}$	$\Theta(\log \log n)$	Rabin's closest pair of points algorithm
$f(n) = \log n$	$\Theta(\log^* n)$	Succinct binary rank

Iterated Logarithms

- ***Intuition:*** The log function is incredibly effective at shrinking down large quantities.
  - Number of protons in the known universe: $\approx 2^{240}$.
  - $\log^{(0)} 2^{240} = 1{,}766{,}847,$[… 57 digits …]$,292{,}619{,}776$
  - $\log^{(1)} 2^{240} = 240$
  - $\log^{(2)} 2^{240} \approx 7.91$
  - $\log^{(3)} 2^{240} \approx 2.98$
  - $\log^{(4)} 2^{240} \approx 1.58$
  - $\log^{(5)} 2^{240} \approx 0.66$
- So $\log^* 2^{240} = 4$.
- The ***iterated logarithm of n***, denoted **log* *n***, grows *much* more slowly than $\log n$.

Intuiting log* n

- What is log* n for the value of n shown below?

$$n = 2^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2}}}}}}}}}}}}}}}$$

- ***Answer***: log* n = 16.
- The value of n is inconceivably large, and yet log* n is small enough to hold in your hand. The log* function grows very, very slowly!

Iterated Iterates

- What is the value of this expression?

$$\log^{**}\left(2^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2}}}}}}}}}}}}}}}\right)$$

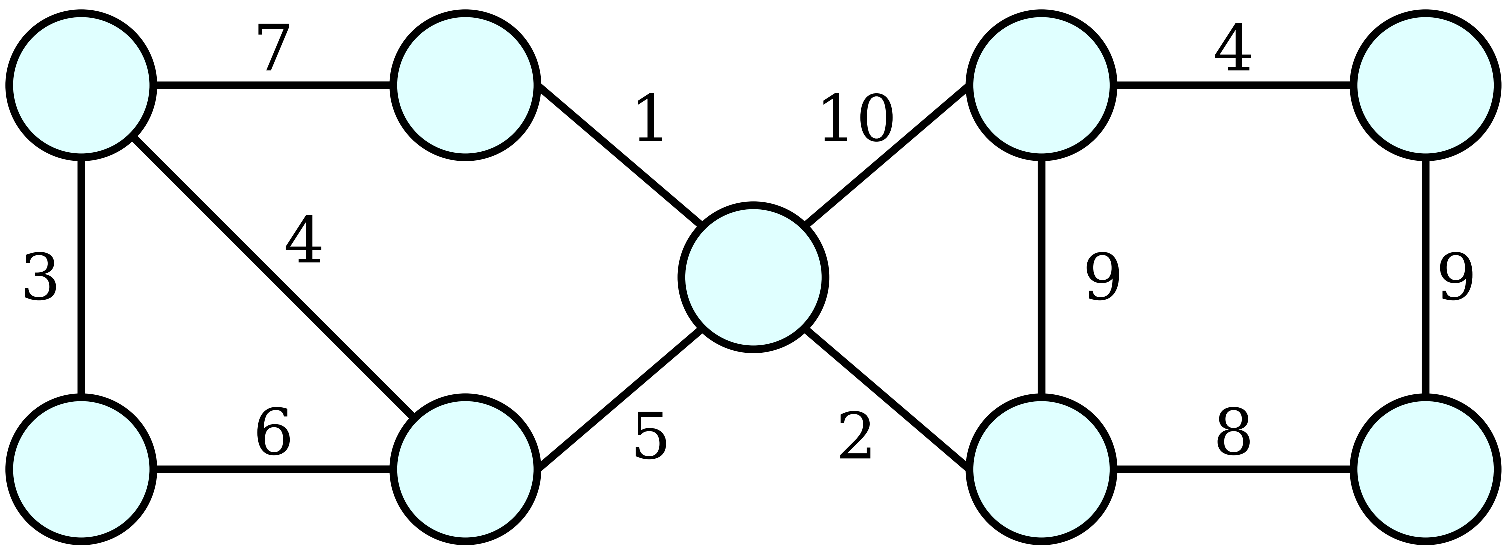
- After taking one log*, we're left with $16 = 2^{2^2}$.
- After taking another log*, we're left with 2.
- After taking another log*, we're left with 0.
- So the above expression evaluates to **2**.
- How big of an input do we need to get log** n to be 3?

Just how slowly can a function grow?

Incremental Dynamic Connectivity

Kruskal’s Algorithm

- ***Kruskal’s Algorithm*** finds an MST of a graph. It works as follows:
  - Remove all edges from the graph and sort them from lowest to highest.
  - Repeatedly insert edges back into the graph, as long as their endpoints aren’t already reachable from each other.



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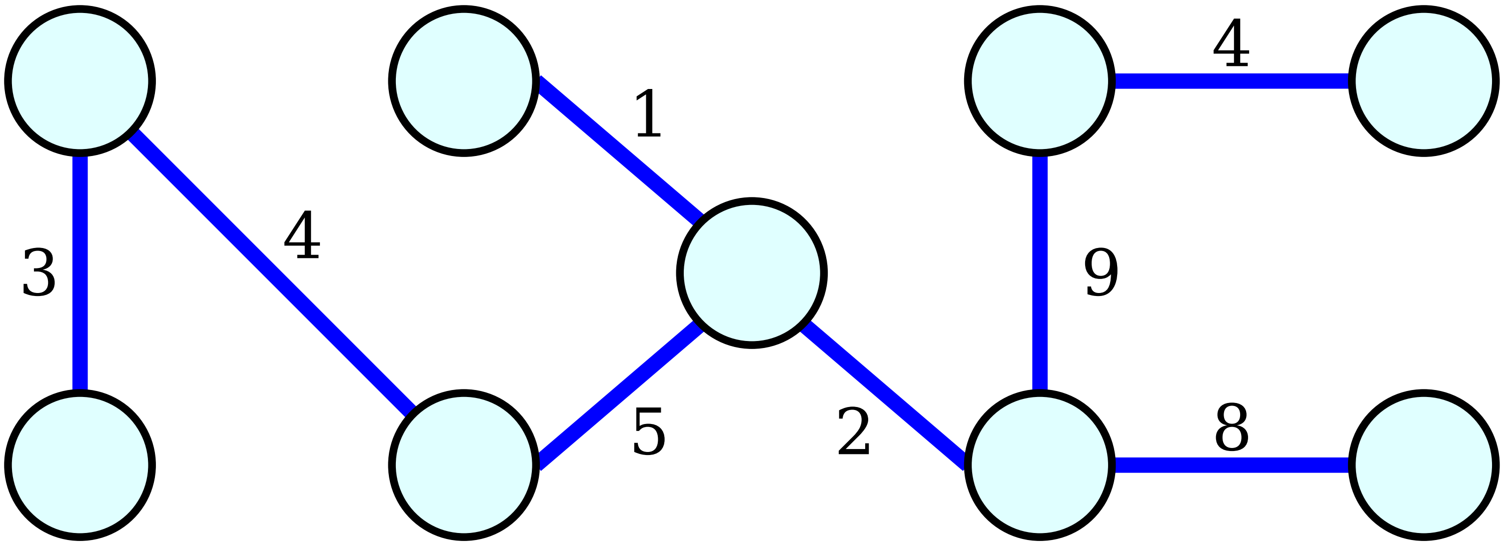
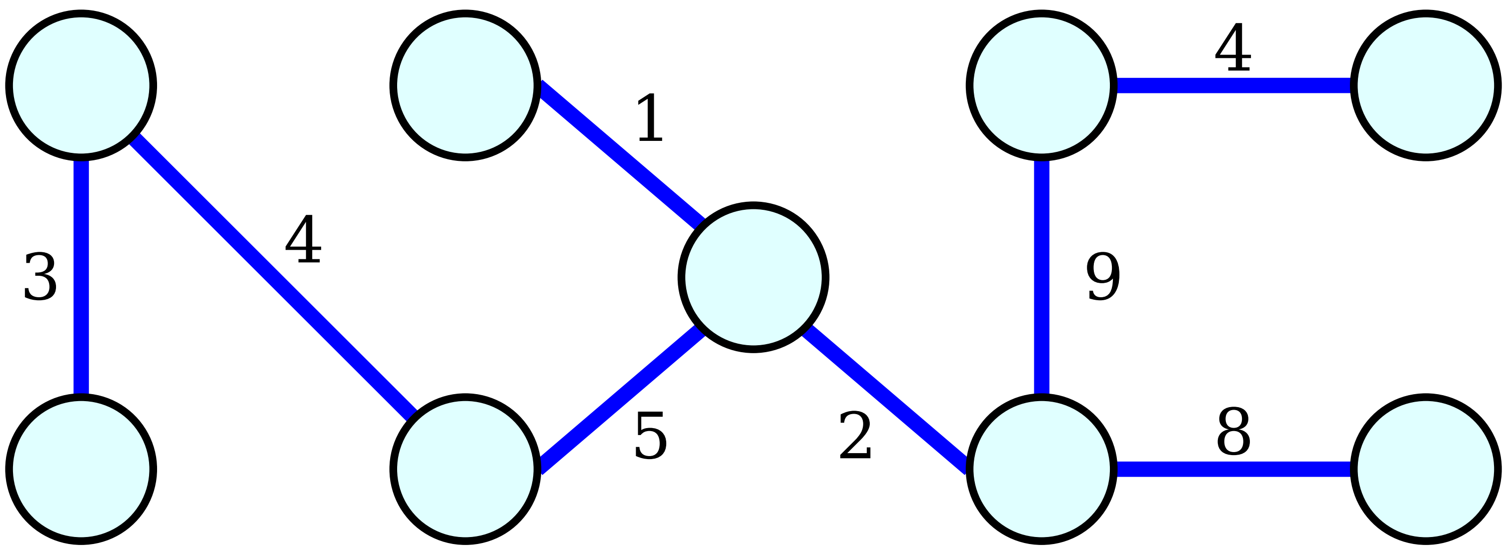


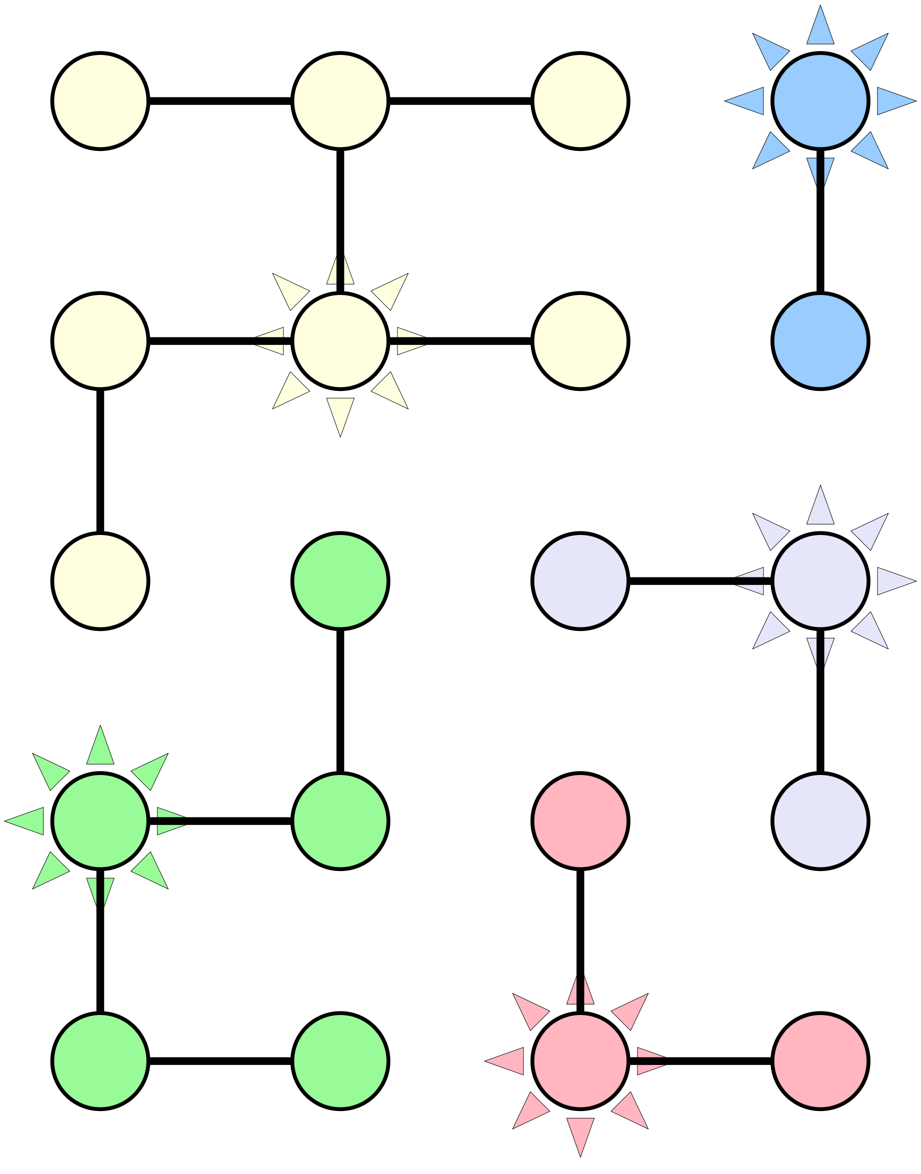
Incremental Connectivity

- Kruskal's algorithm needs a data structure that solves ***incremental connectivity***.
  - We begin with an empty graph.
  - We need to be able to add new edges to the graph and check whether arbitrary pairs of nodes are connected.
- ***Question:*** How efficiently can we do this?



Representatives

- ***Idea:*** Assign a ***representative*** to each CC in the graph.
- To see if two nodes are in the same CC, check if they have the same representative.
- To link together two different CCs, change the representative of all the nodes in one CC to be the representative of the other CC.



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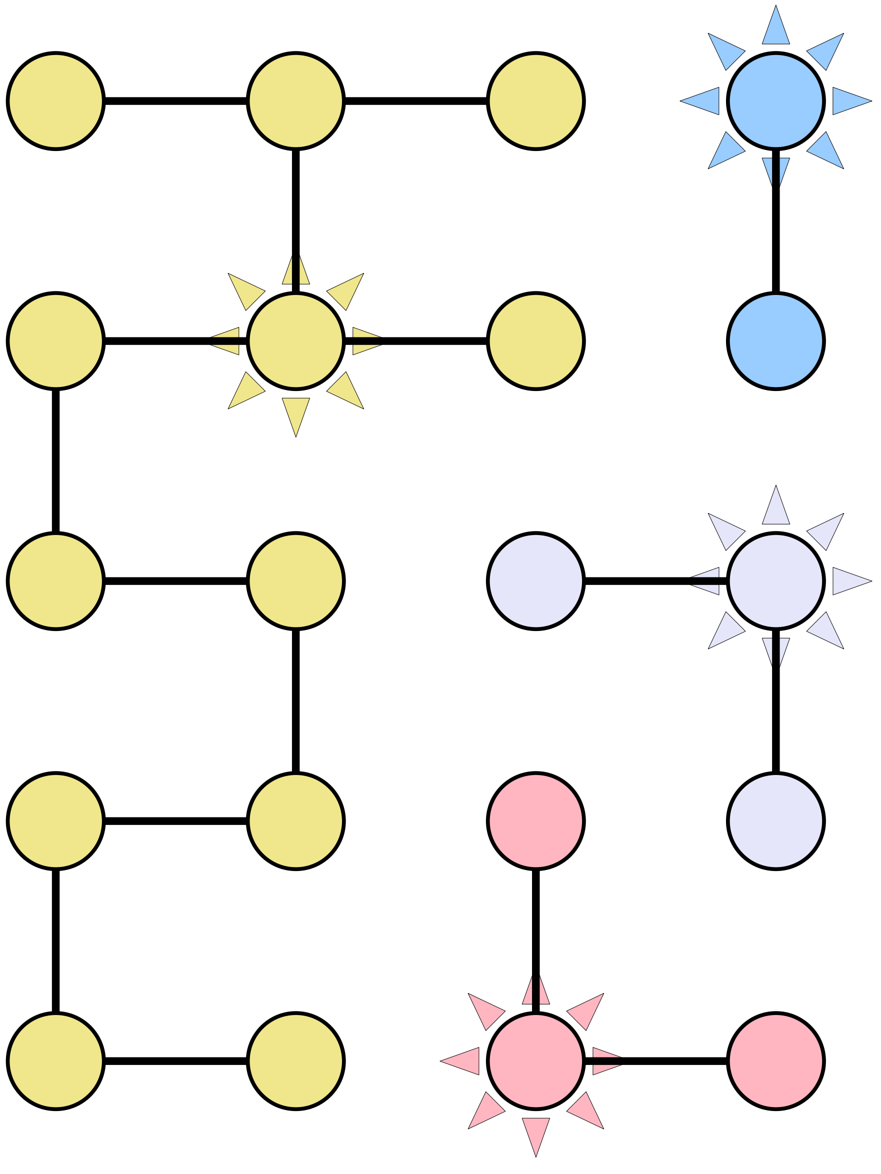
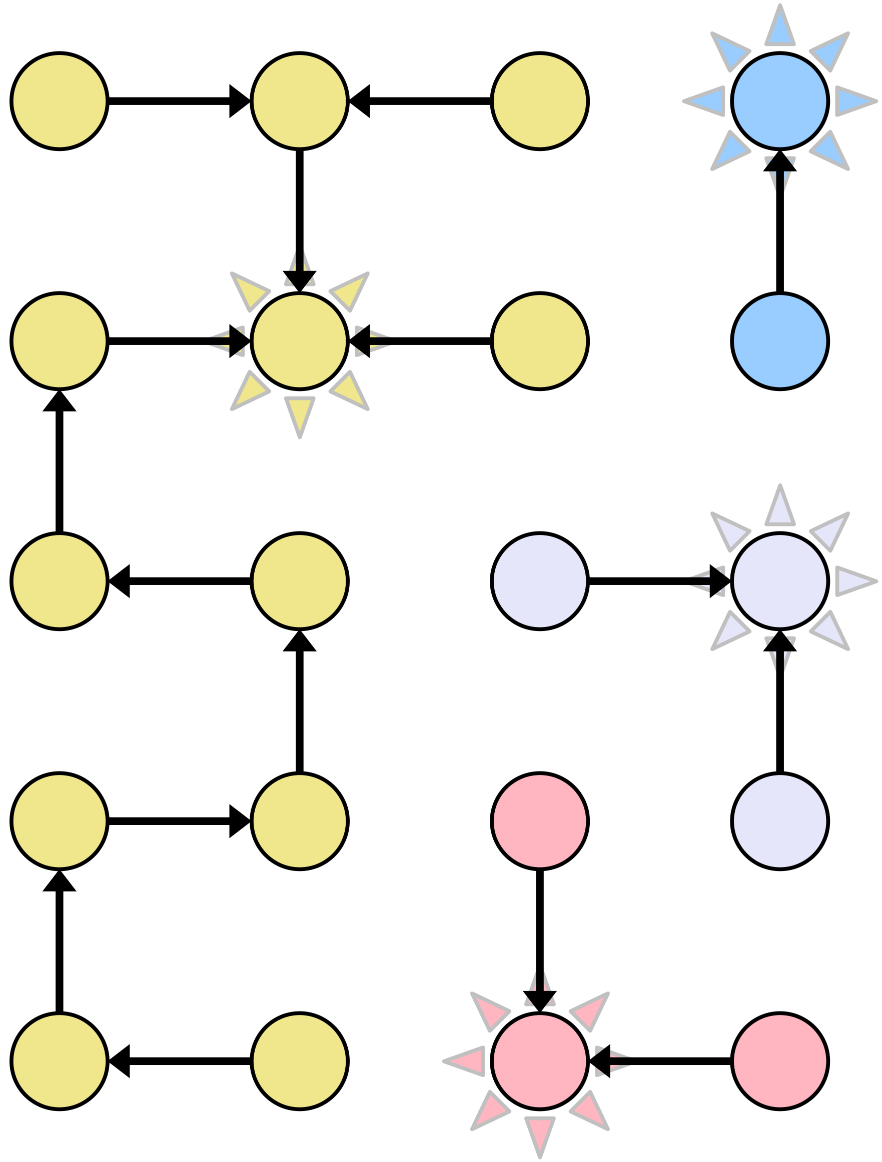


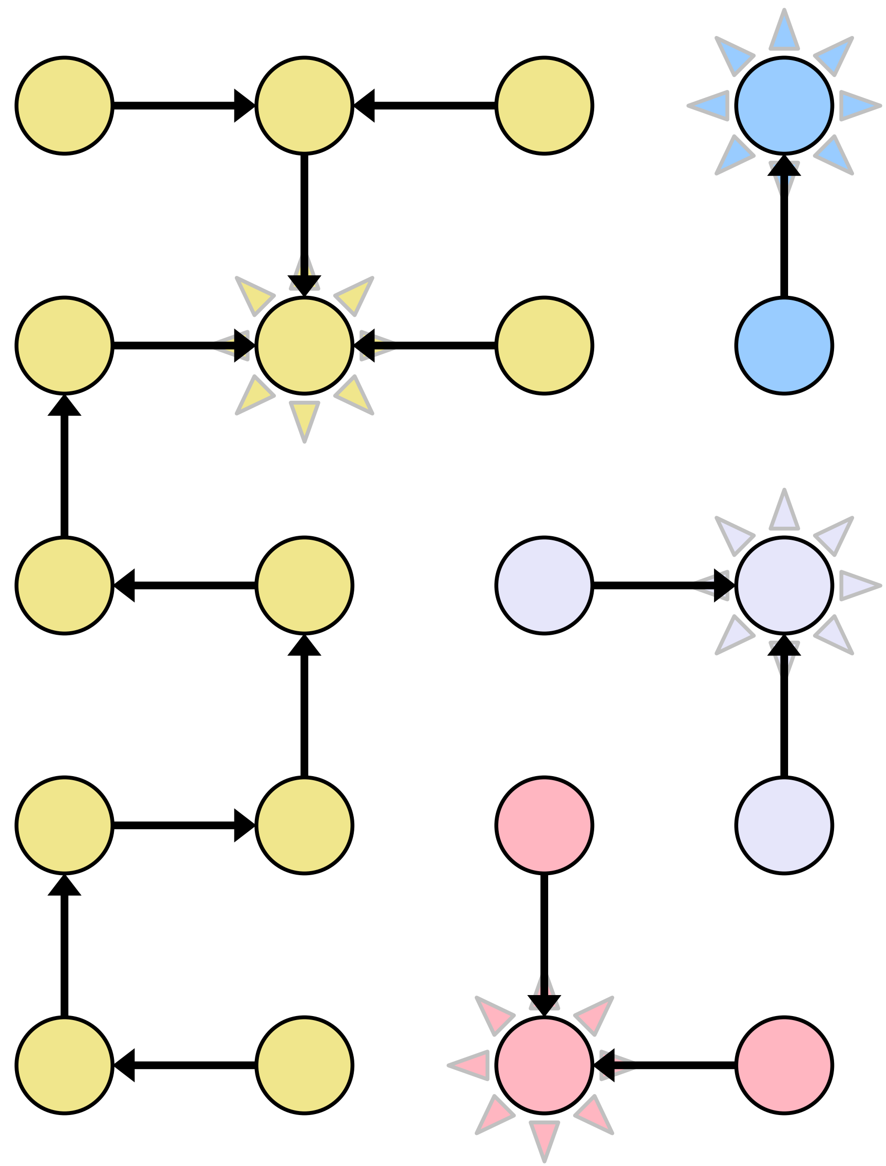
Representatives

- Here's how we'll implement this idea.
  - Each node has a ***parent pointer***.
  - Representatives' parent pointers are null.
  - Other nodes' parent pointers form chains leading to the representative.
- Although the original graph is undirected, parent pointers are directed.
- This data structure is called a ***disjoint-set forest***.



Representatives

- We'll support two operations.
- ***find***(x) returns x's representative. It works by following parent pointers until we hit the representative.
- ***union***(x, y) merges the clusters containing x and y. It works by finding x and y's representatives. If they aren't equal, it assigns one of those representatives the other as a parent.



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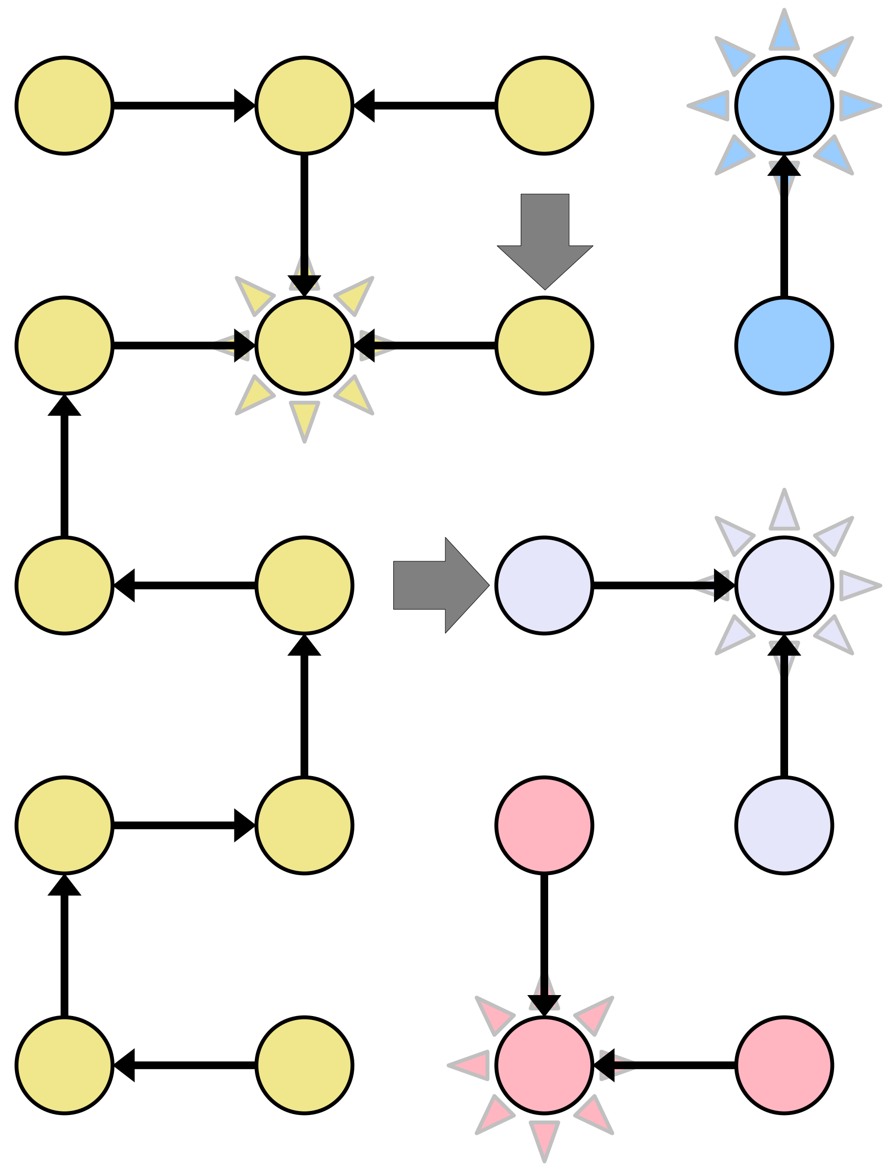

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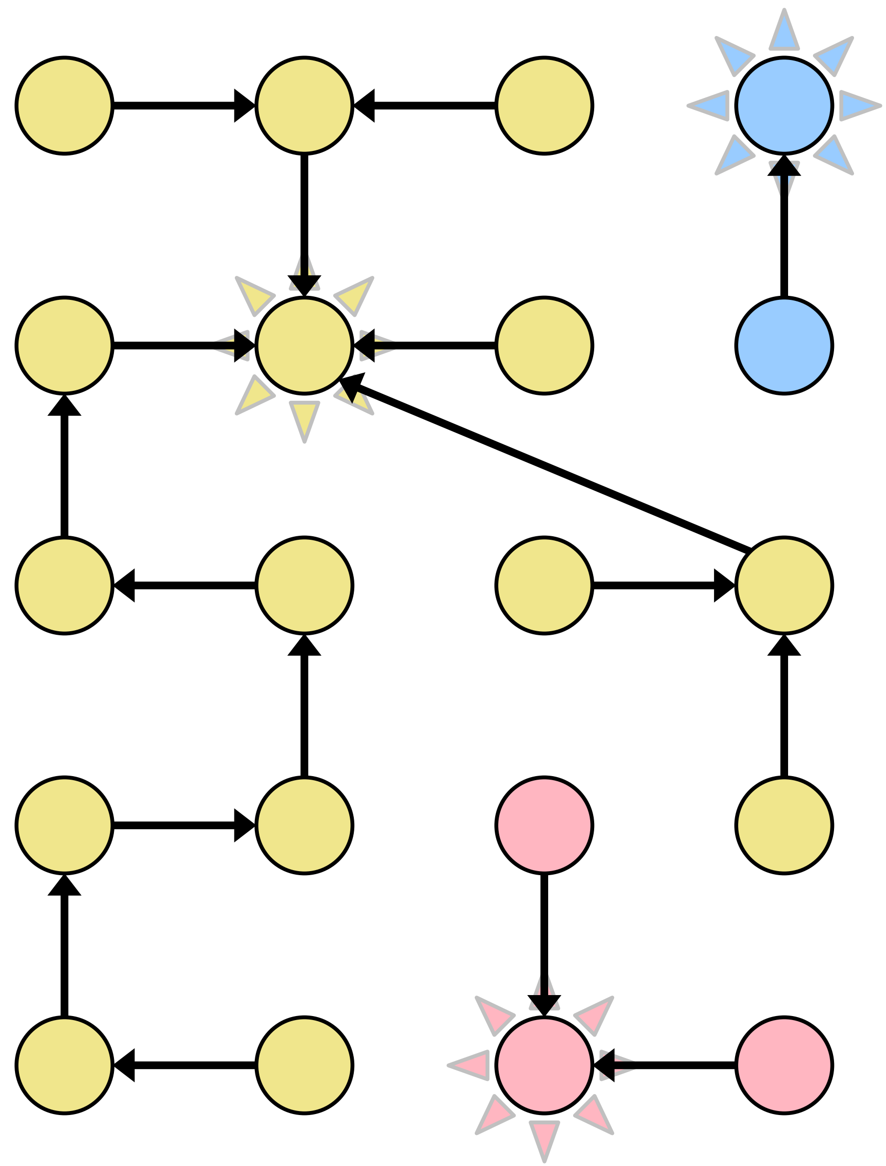

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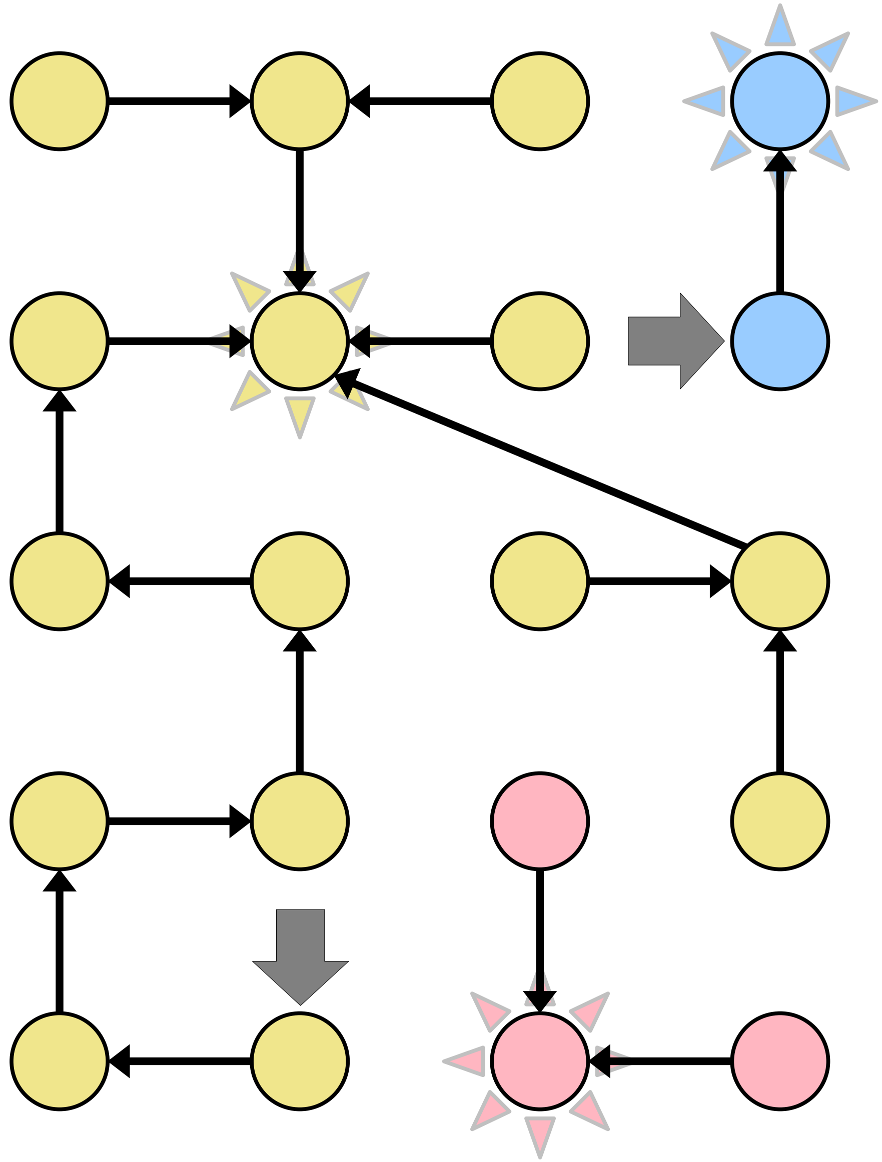

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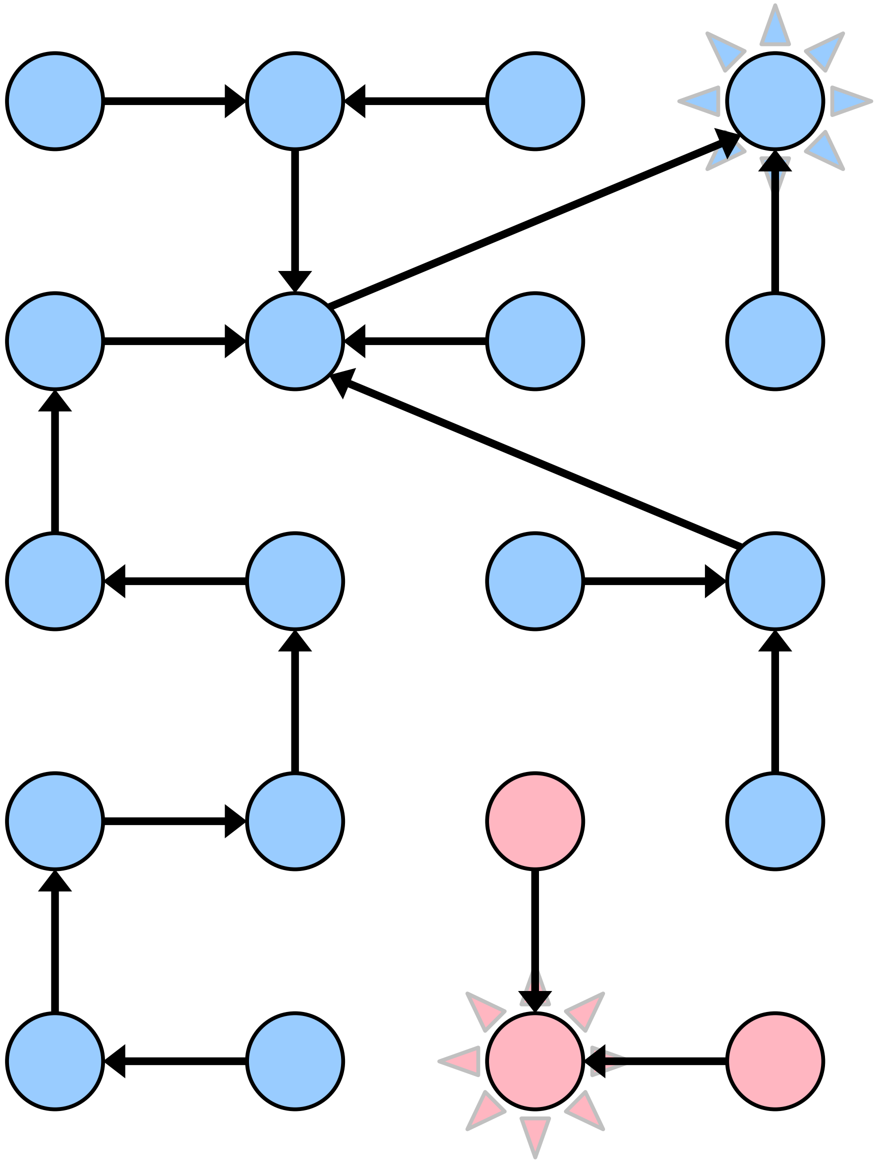
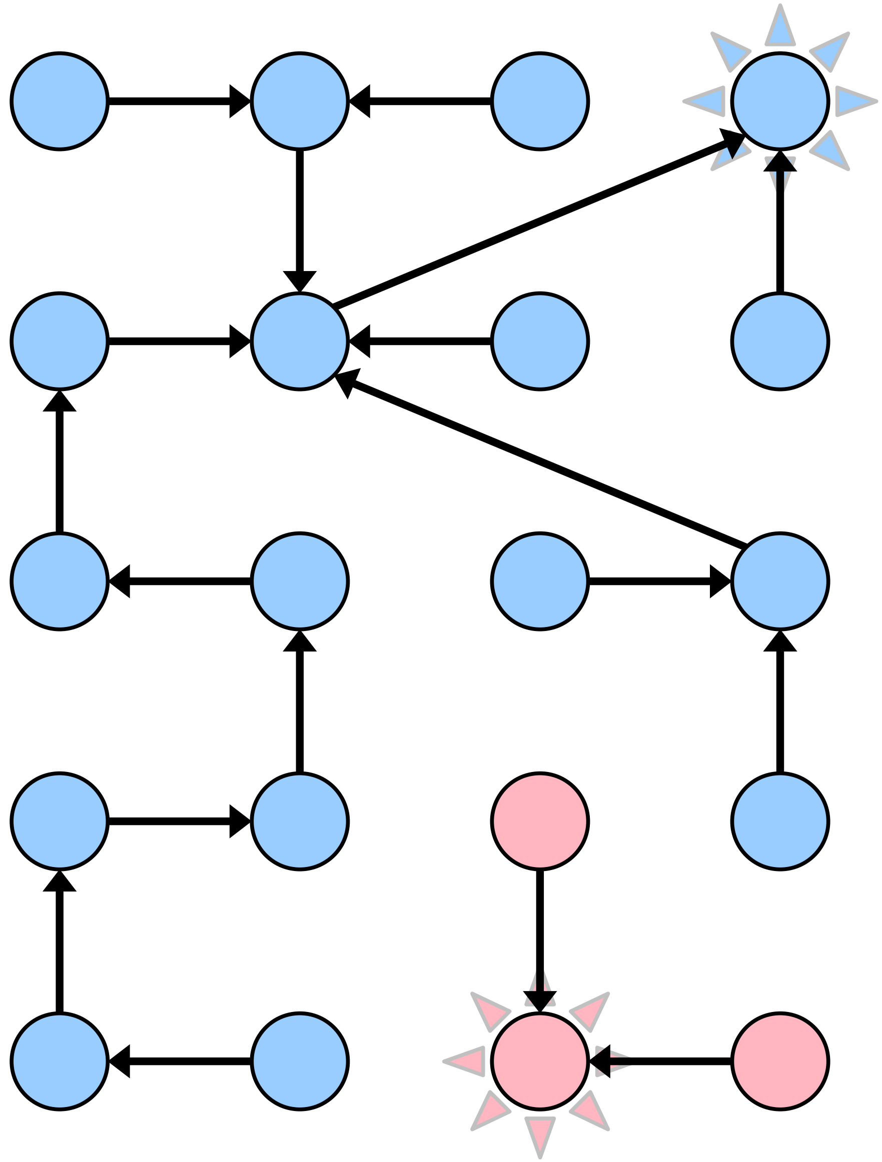


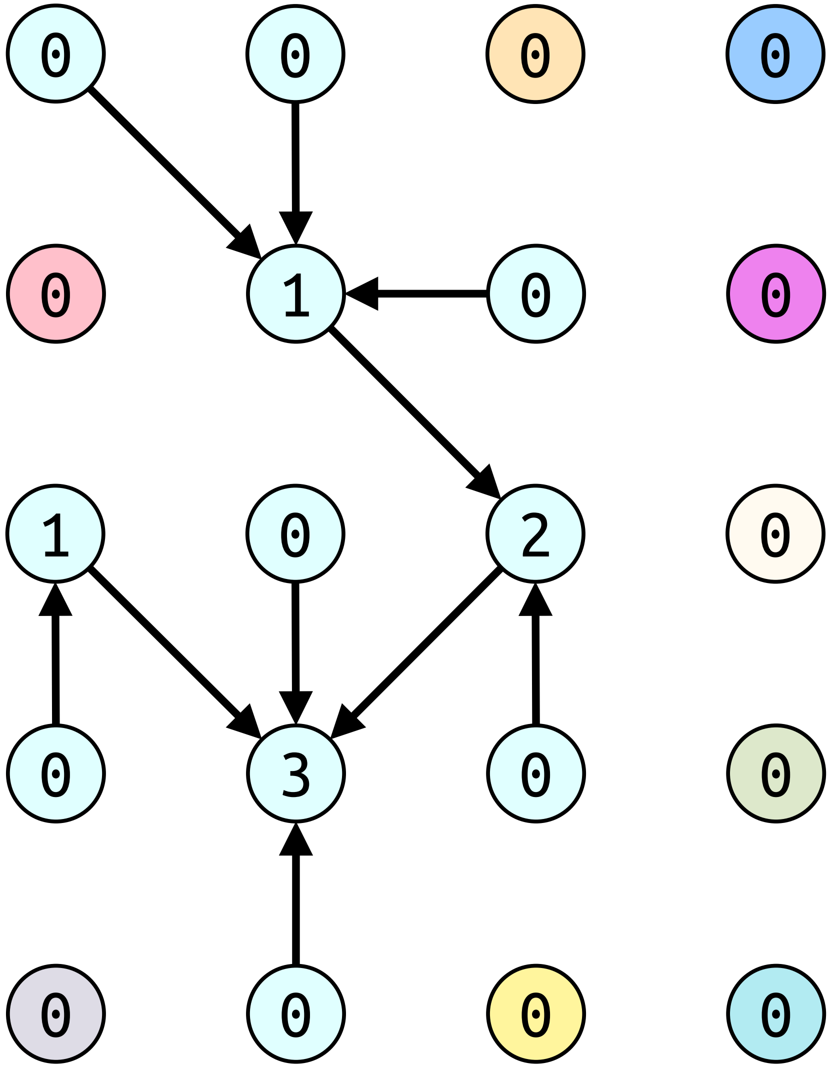
Representatives

- Unfortunately, this system can be very slow.
- If we aren't careful with how we link trees, the cost of a ***find*** or ***union*** can grow to $\Theta(n)$, where n is the number of nodes in the graph.
- ***Can we do better?***



Union-By-Rank

- Assign each node a ***rank***, initially 0.
- When linking two representatives x and y:
    - If one representative has a lower rank than the other, set its parent to the other.
    - Otherwise, arbitrarily set x's parent to y, then increment y's rank.
- This keeps the lengths of parent chains low.

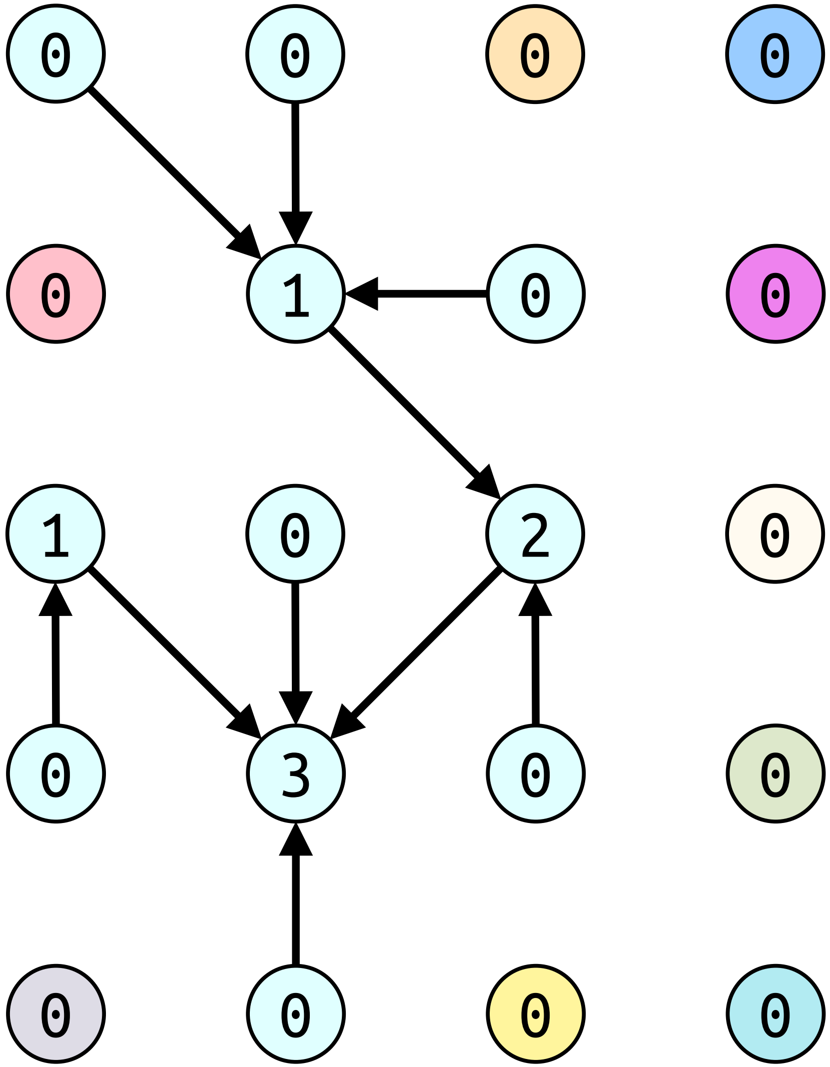


Union-By-Rank

- ***Lemma:*** A node of rank *r* has children of ranks 0, 1, 2, …, and *r* – 1.

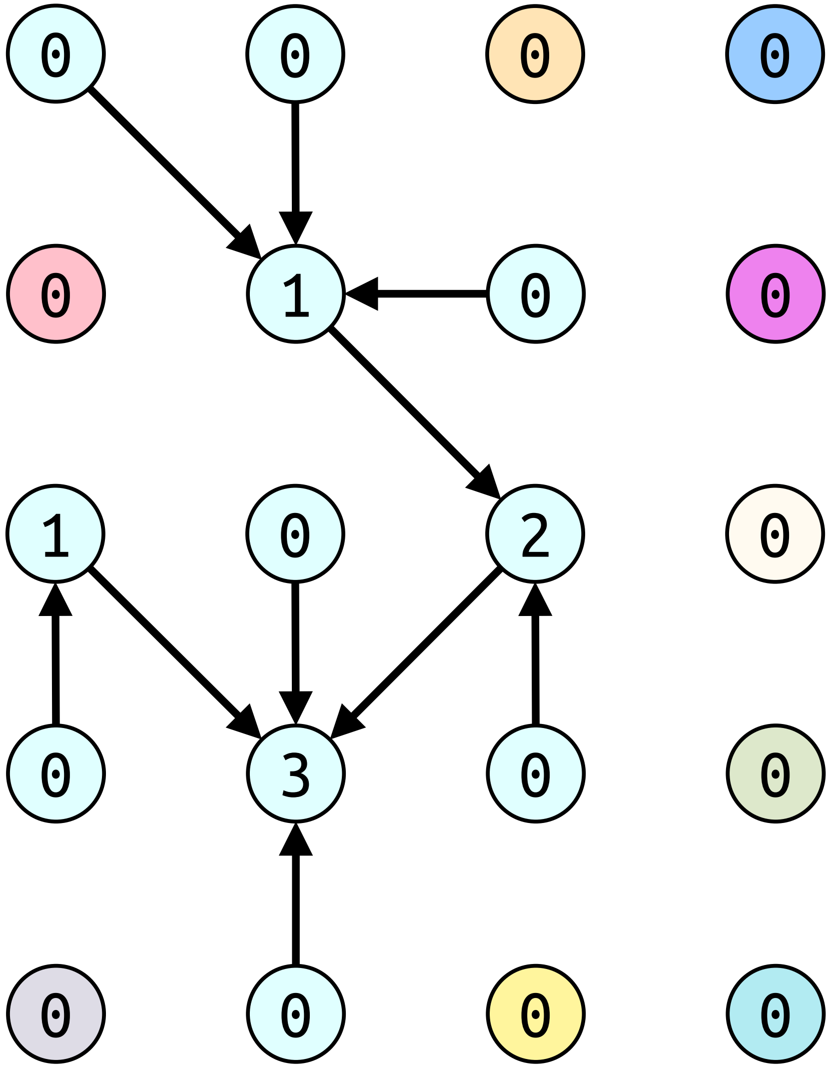
Why?

Answer at
https://pollev.com/cs166spr23



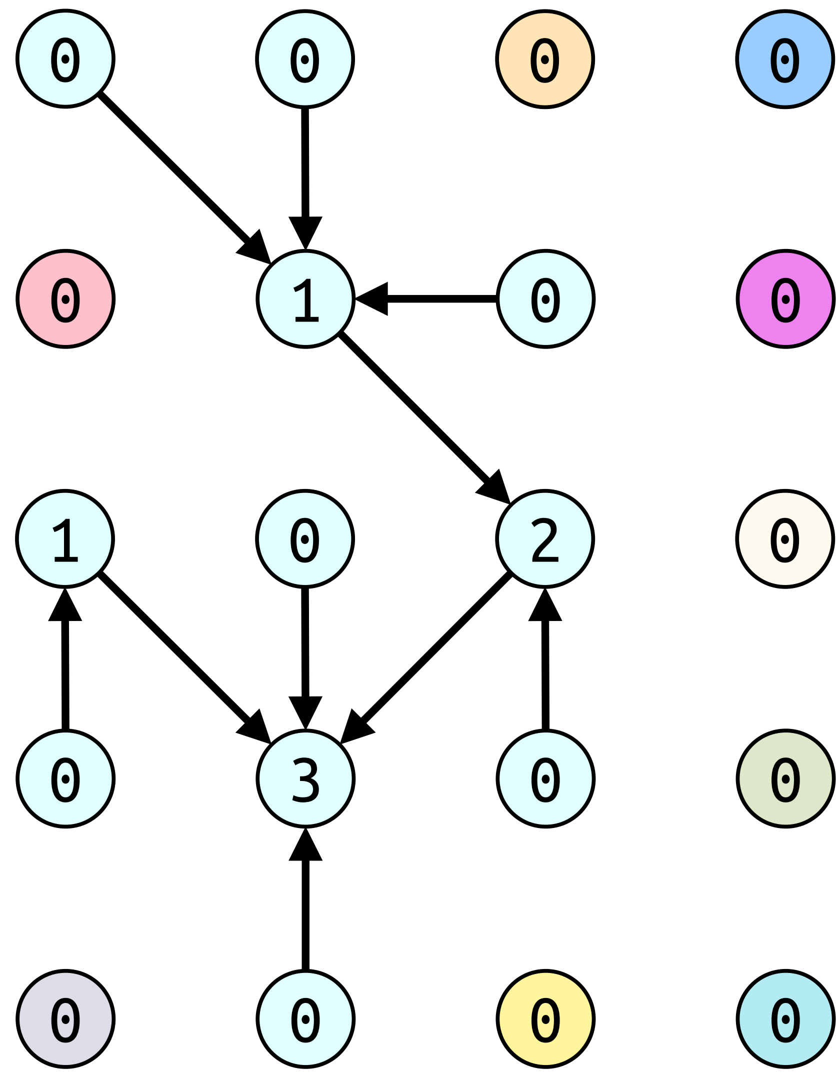
Union-By-Rank

- ***Lemma:*** A node of rank r has children of ranks $0, 1, 2, \ldots,$ and $r - 1$.
- ***Proof:*** Induction!
  - A node of rank 0 has no children.
  - A node v of rank $r + 1$, at the time its rank was increased, was a tree of rank r that got another tree of rank r as a child.
  - By the IH v already had children of ranks $0, 1, 2, \ldots, r - 1$. Now it also has a child of rank r. ■



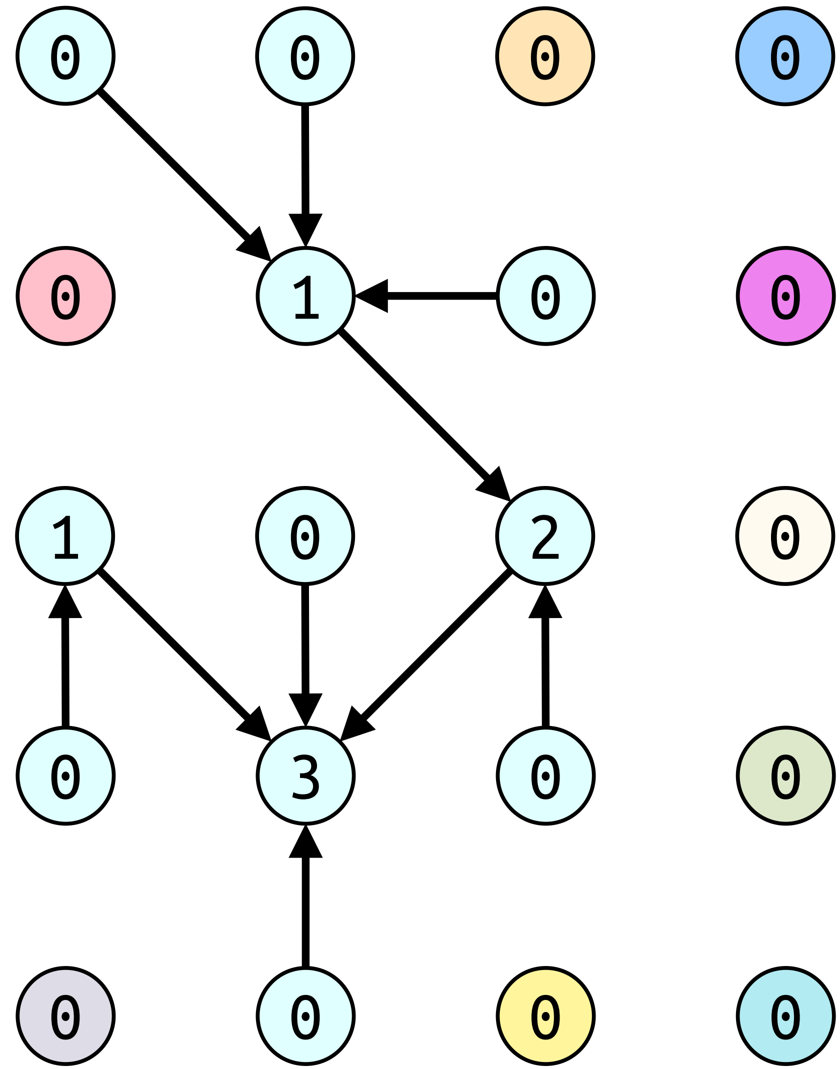
Union-By-Rank

- Our lemma tells us, indirectly, that the "simplest" tree whose root has rank *r* is a binomial tree of order *r*.
- A nice consequence of this is that all trees in a forest of *n* nodes have height O(log *n*), so each ***union*** and ***find*** takes time O(log *n*).
- ***Can we do better?***



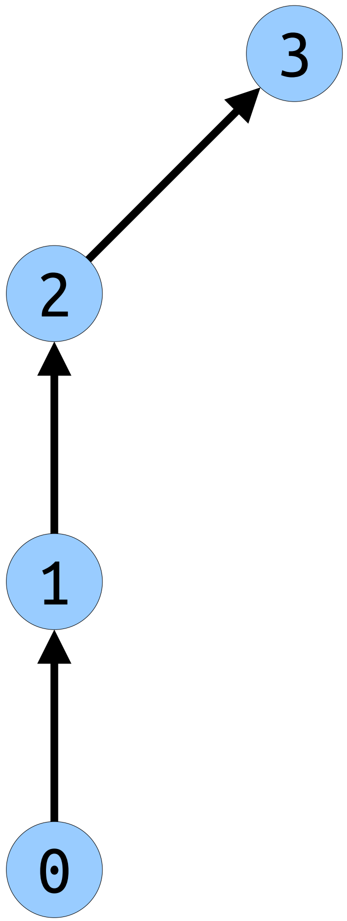
An Observation

- Suppose we call ***find***(x) multiple times.
- Each time we do that, we may have to traverse a chain of O(log n) nodes to find its representative.
- Do we really need to scan things so many times?



Path Compression

- ***Path compression*** is an optimization on the ***find*** operation.
- After figuring out *x*'s representative, change the parent pointers of all of *x*'s ancestors to point directly to *x*'s representative.
- This makes it a lot faster to find representatives across multiple operations.



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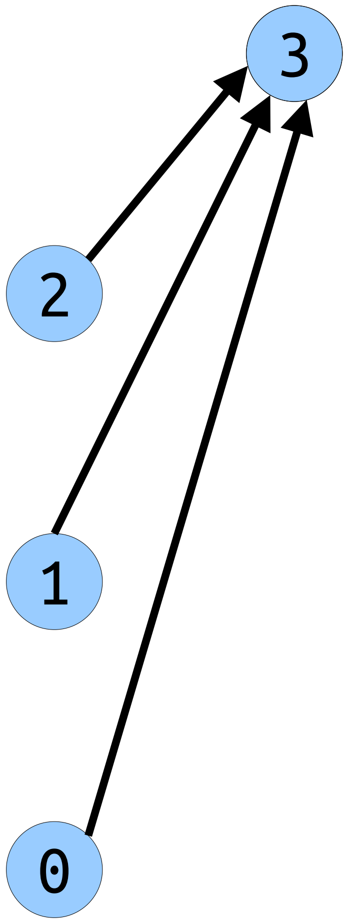


Path Compression

- The resulting code for our data structure is surprisingly simple:

```
Node* find(Node* source) {
    if (source->parent == nullptr) return source;

    /* Path compression: update parent before returning. */
    source->parent = find(source→parent);
    return source->parent;
}

void doUnion(Node* one, Node* two) {
    /* Find the representatives. */
    one = find(one);
    two = find(two);
    if (one->rank > two->rank) swap(one, two);

    /* Link and update ranks if needed. */
    one->parent = two;
    if (one->rank == two->rank) two->rank++;
}
```

- Now, all we have to do is analyze the runtime.

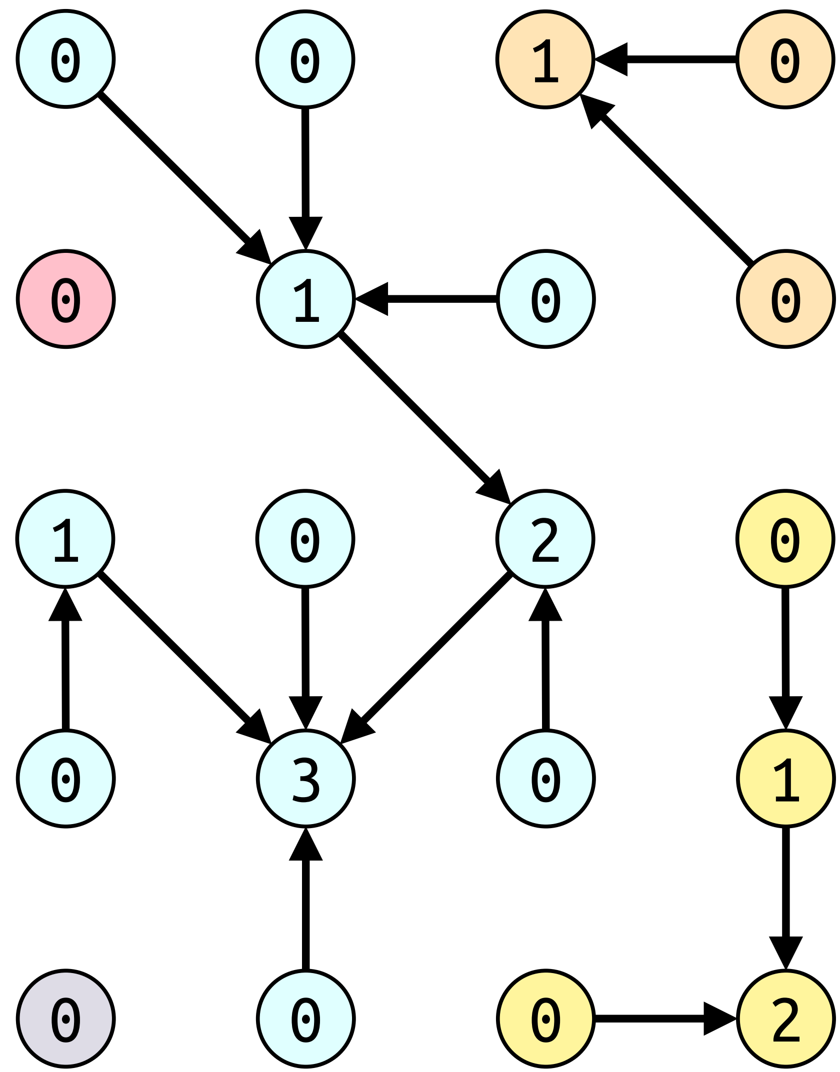
Analyzing Disjoint-Set Forests

History

- The analysis of union-by-rank plus path compression has a long history.
- For a while, its actual efficiency was an open problem!
- In 1979 Tarjan proved a tight upper bound on the runtime using a clever and nuanced analysis, and provided a matching lower bound.
- In 2003 Seidel and Sharir arrived at the same upper bound using a totally different technique.
- Both analyses require a careful analysis of the costs of the operations and result in a very surprising result.
- The analysis I'll share comes from Seidel and Sharir and is based on ***this set of lecture slides*** from an algorithms course at Harvard.

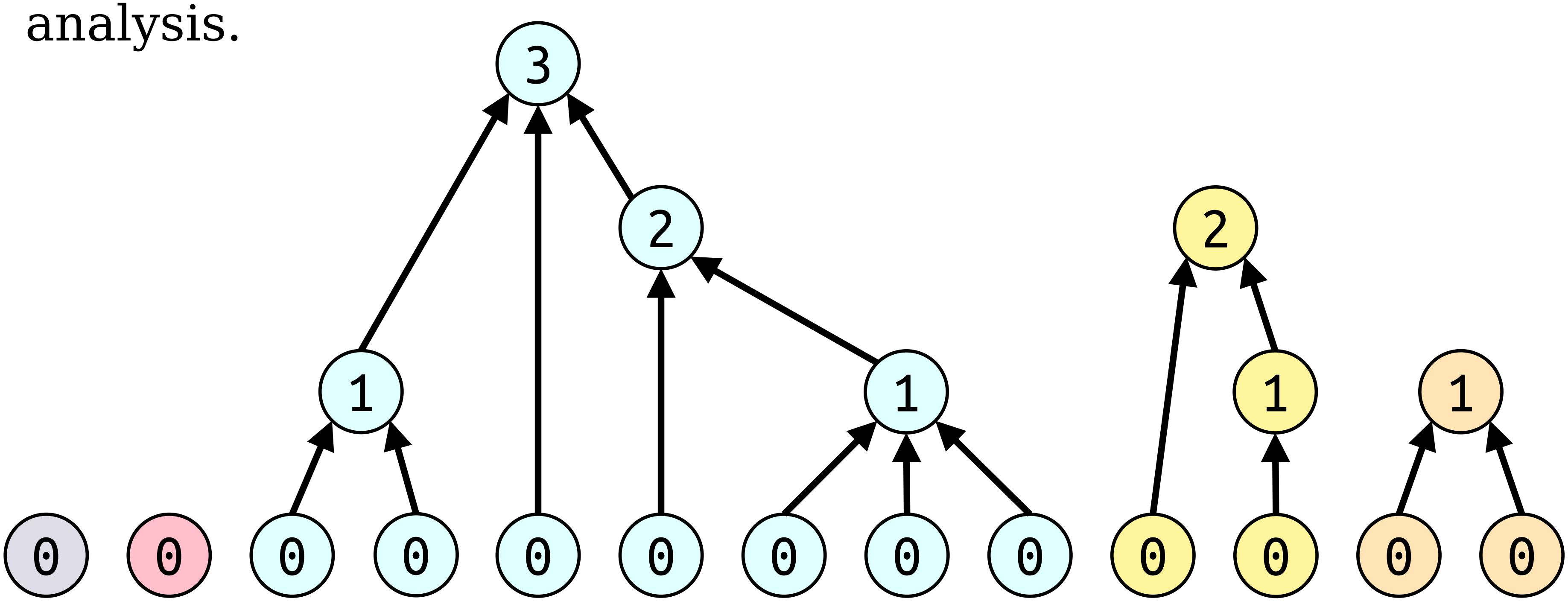
Our Analysis

- We're going to analyze a slightly simplified version of this problem.
- We'll be given a forest $\mathscr{F}$ formed purely from union-by-rank, then do a series of path compressions on it.
  - These don't have to go all the way from a node to its representative.
- Our goal will be to bound the total amount of work done.
- ***Great Exercise:*** Show that this analysis carries over to the case of interleaved ***union***s and ***find***s.



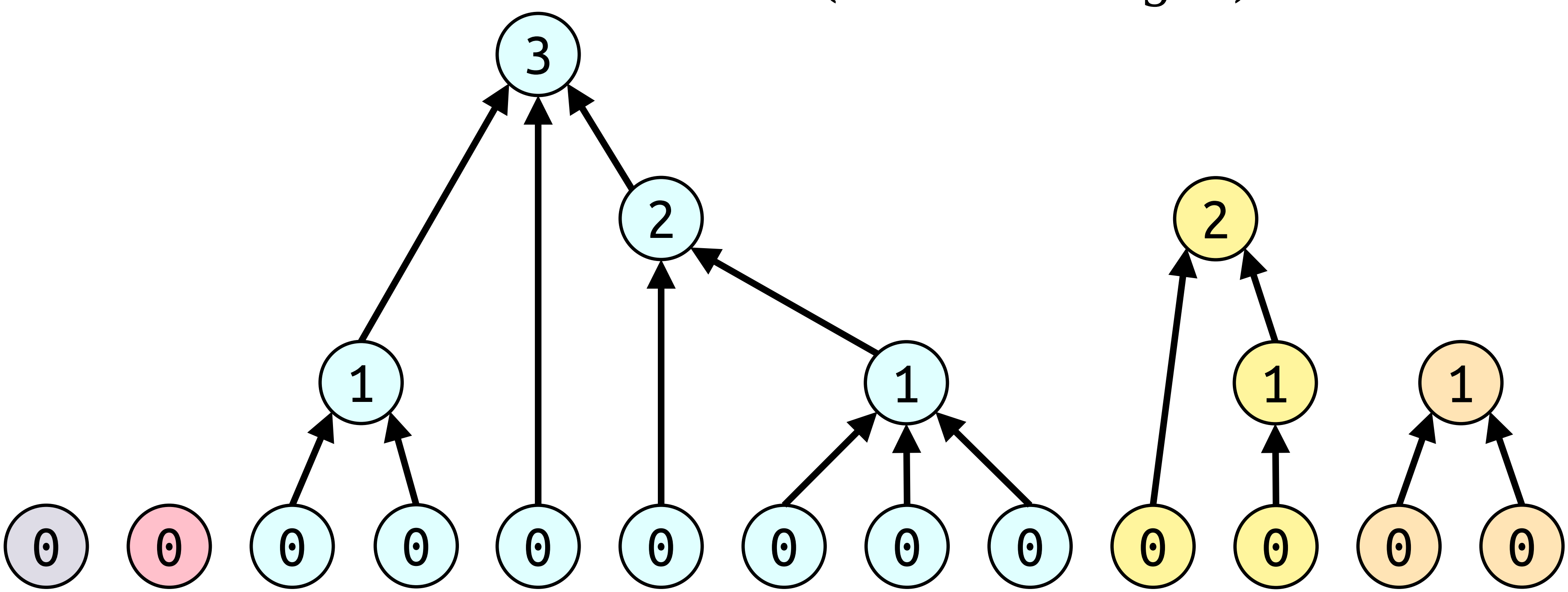
Our Analysis

- Some notation we'll use throughout this analysis:
  - Let $\boldsymbol{n}$ be the number of nodes in the disjoint-set forest.
  - Let $\boldsymbol{m}$ be the number of operations performed.
  - Let $\boldsymbol{r}$ be the maximum rank of any node in the forest. (We know $r = O(\log n)$, but could be lower.)
- In practice, we'll have $\boldsymbol{m = \Omega(n)}$, and we'll assume this in our analysis.



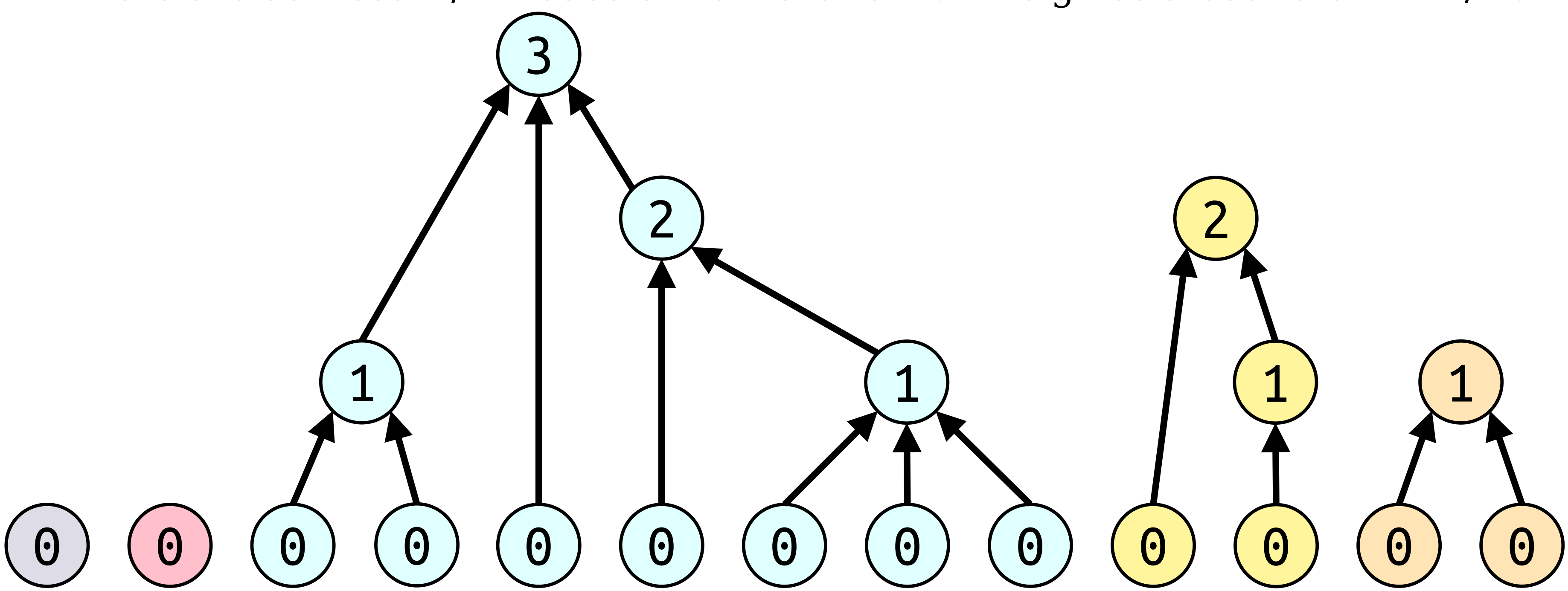
Our Analysis

- We will specifically focus on the number of times a node's parent changes.
- Why?
  - Each operation does O(1) work, plus work proportional to the number of parents changed.
- The total work done is then $\Theta(m + \#\text{changes})$.



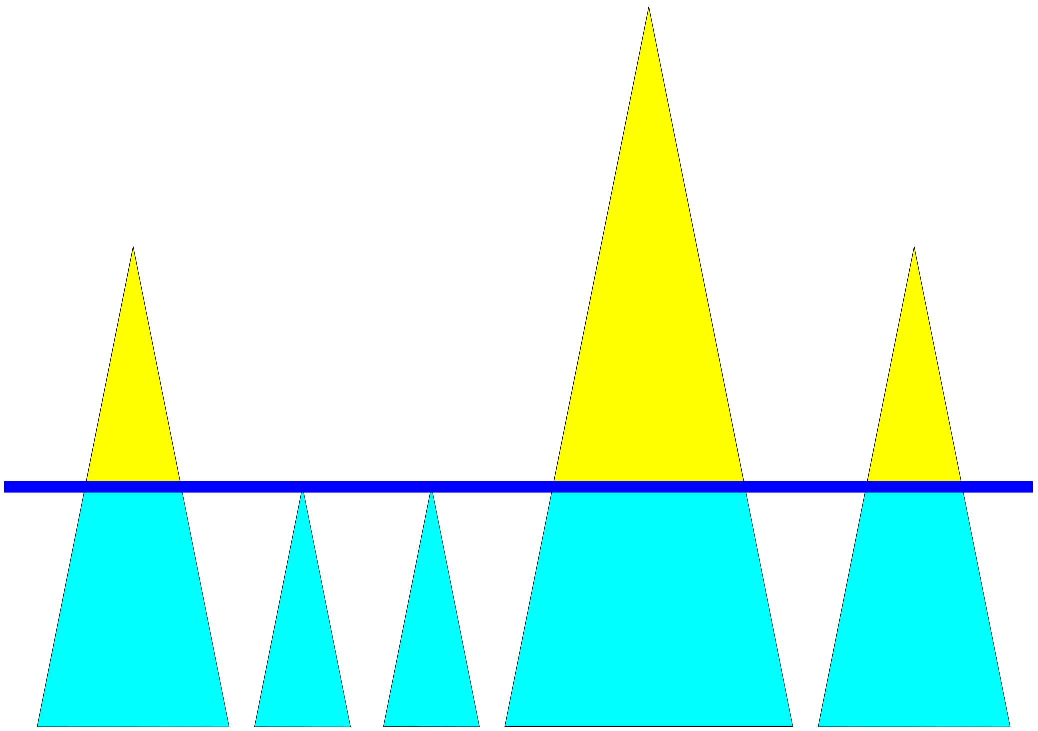
A Starting Analysis

- ***Lemma:*** The number of pointer changes is at most $m + n \cdot r / 2$.
- ***Proof Sketch:*** Consider nodes of zero and nonzero rank.
  - ***Nodes of rank 0:*** A node of rank 0 only has its parent change if it is the start node of a compress. There are m compresses, so these pointers change at most m times.
  - ***Nodes of nonzero rank:*** When a parent changes, the new parent's rank is bigger than the old parent's rank, so a node's rank can increase at most r times. There are at most $n / 2$ nodes of nonzero rank. This gives a bound of $n \cdot r / 2$.



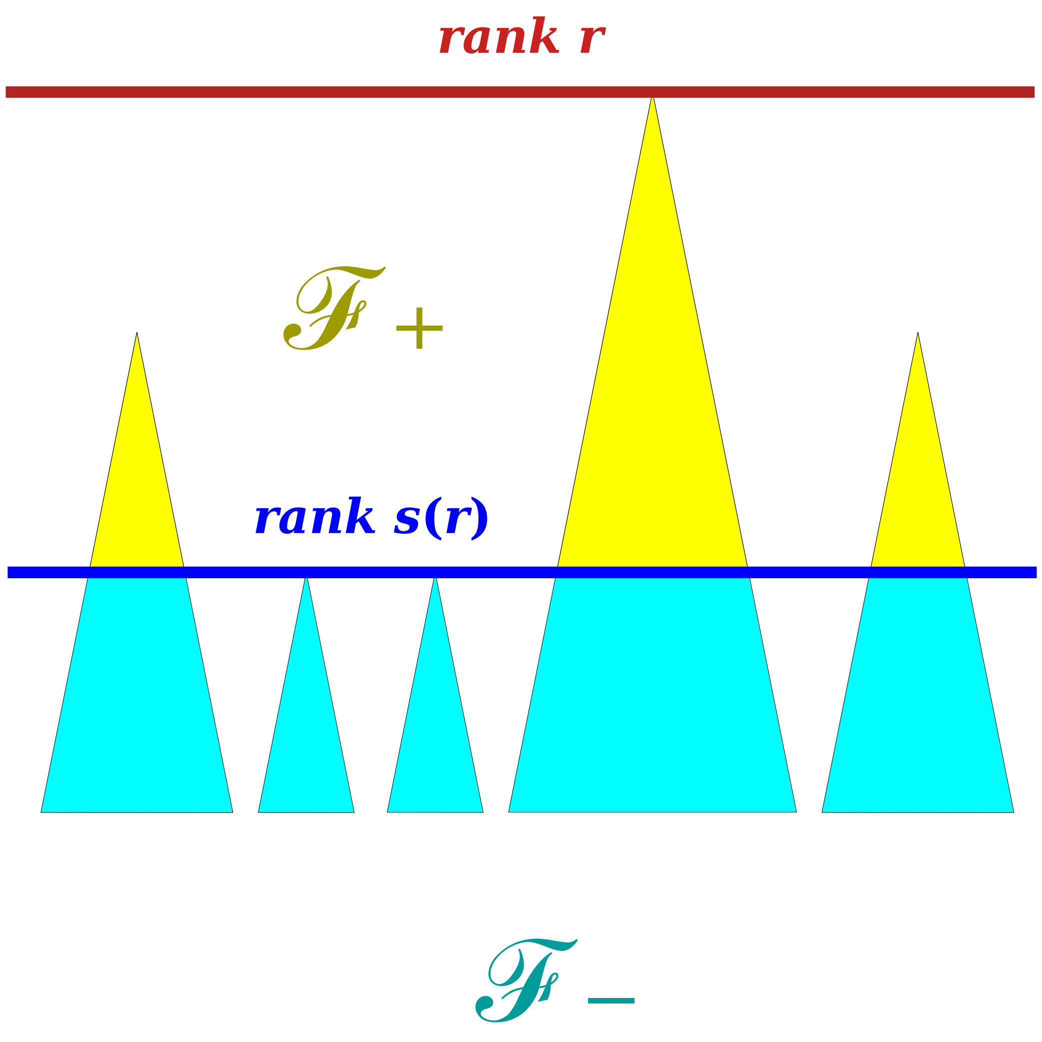
A Starting Analysis

- Our starting analysis is weak.
  - Compressing a path impacts other nodes not on that path.
  - Nodes with high starting rank have can't have their parents change too many times.
- These effects work differently in different parts of the tree.
  - The first effect is more pronounced at the bottom of the forest.
  - The second effect is more pronounced at the top.
- ***Idea:*** Split the forest into a "top forest" and "bottom forest," and analyze the costs in each forest separately.



Forest Slicing

- As before, let r be the maximum rank in $\mathscr{F}$.
- Suppose that, somehow, we pick a rank $s(r)$ as a separating rank.
- Then, split our forest $\mathscr{F}$ into two forests:
  - $\mathscr{F}_-$ consists of all nodes of rank $s(r)$ or below.
  - $\mathscr{F}_+$ consists of all nodes of rank above $s(r)$.
- ***Goal:*** Split the cost of compressions across $\mathscr{F}_-$ and $\mathscr{F}_+$.



Some Terminology

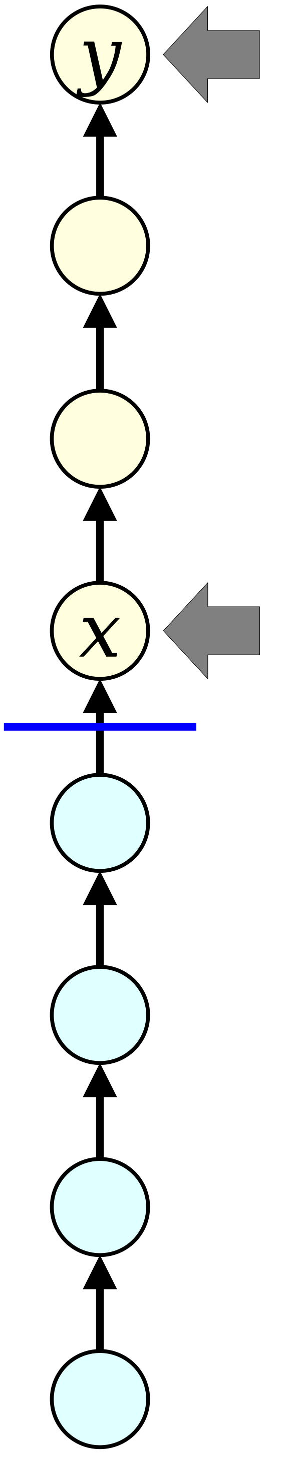
- Let ***C*(*m*, *n*, *r*)** be the maximum number of pointer changes that can be made if there are m compresses, n nodes, and the maximum rank is r.
- Using this notation, our earlier result is that

$$C(m, n, r) \leq m + n \cdot r \,/\, 2.$$

- ***Question:*** What is $C(m, n, 0)$? What's $C(m, n, 1)$?
- ***Goal:*** Split $\mathscr{F}$ into $\mathscr{F}_+$ and $\mathscr{F}_-$, find a way to write a recurrence relation for $C(m, n, r)$, then solve the recurrence to get a tight bound on the cost of any series of ***unions*** and ***finds***.

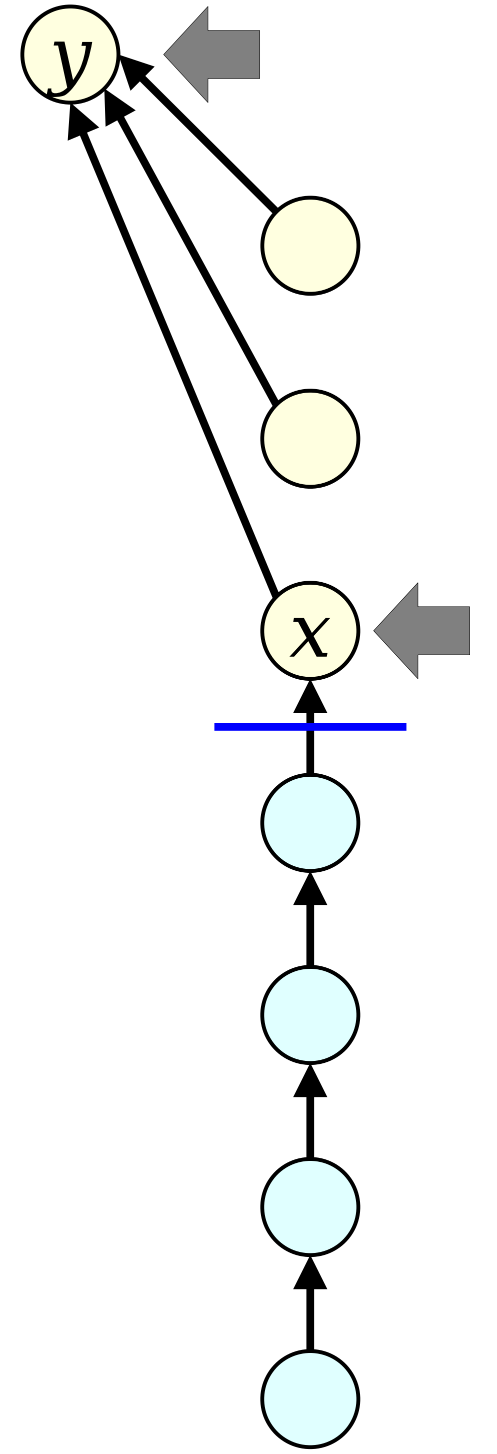
Forest Slicing

- Focus on any one compression from x to y. Let's see how it interacts with $\mathscr{F}_-$ and $\mathscr{F}_+$.
- ***Case 1:*** x and y are both in $\mathscr{F}_+$.



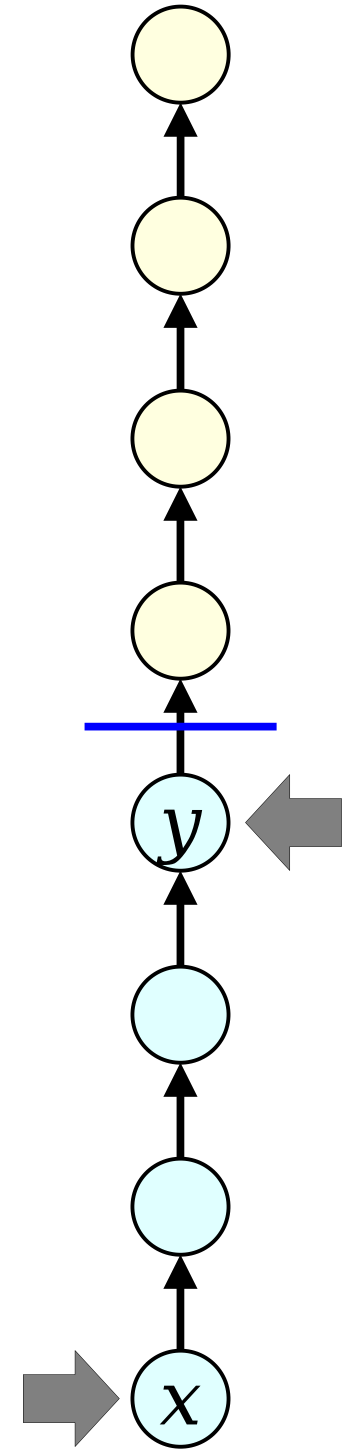
Forest Slicing

- Focus on any one compression from x to y. Let's see how it interacts with $\mathscr{F}_-$ and $\mathscr{F}_+$.
- ***Case 1:*** x and y are both in $\mathscr{F}_+$.
- We can recursively handle this compression when bounding the work done purely in $\mathscr{F}_+$.



Forest Slicing

- ***Case 2:*** x and y are both in $\mathscr{F}_-$.



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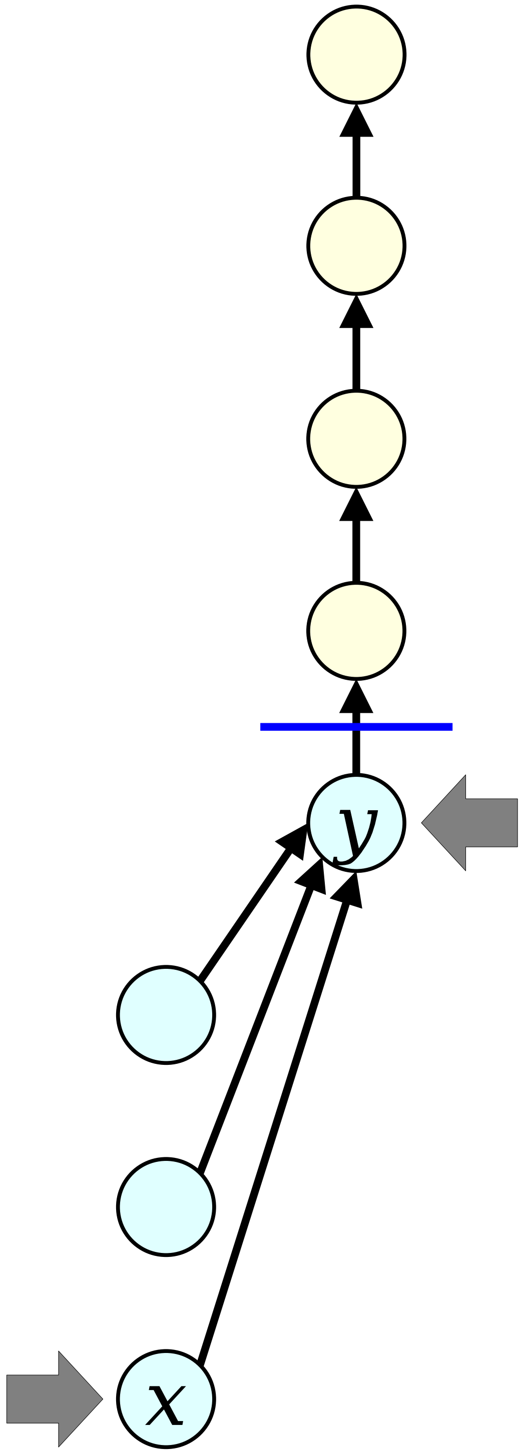
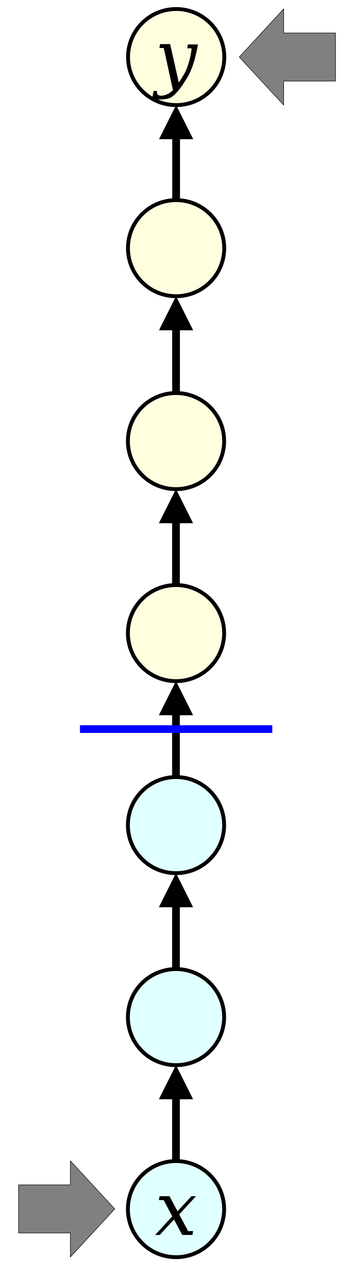


Forest Slicing

- ***Case 3:*** x is in $\mathscr{F}_-$ and y is in $\mathscr{F}_+$.



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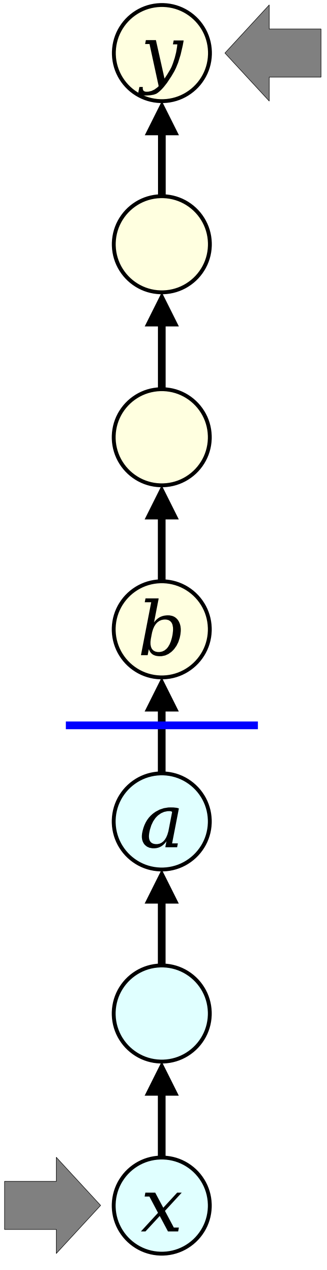
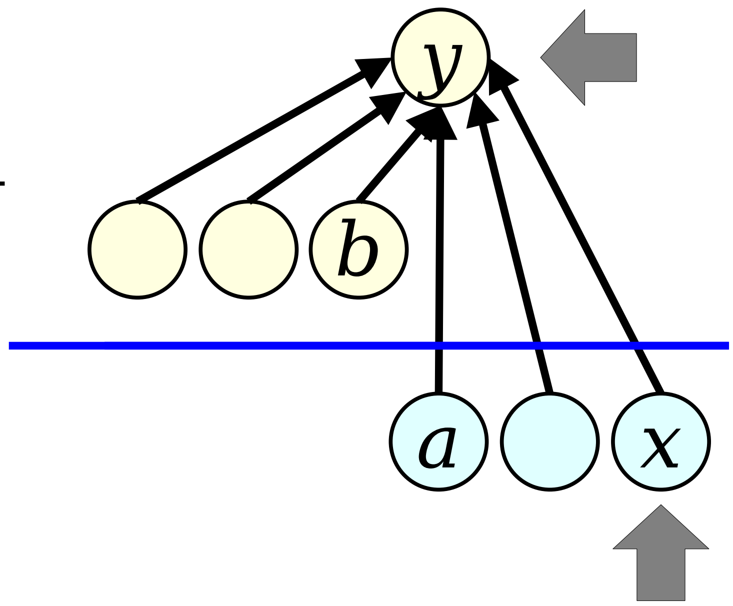


Forest Slicing

- ***Case 3:*** x is in $\mathscr{F}_-$ and y is in $\mathscr{F}_+$.
- We compress from b to y, purely in $\mathscr{F}_+$.
- a, whose parent was already in $\mathscr{F}_+$, gets a new parent in $\mathscr{F}_+$.
- Every node from x (inclusive) and a (exclusive), whose parents were in $\mathscr{F}_-$, now has a parent in $\mathscr{F}_+$.



Forest Slicing

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- Every node from x (inclusive) and a (exclusive), whose parents were in $\mathscr{F}_-$, now has a parent in $\mathscr{F}_+$.

Recursively handle this when processing $\mathscr{F}_+$.

Happens once per compression from $\mathscr{F}_-$ to $\mathscr{F}_+$.

Happens once per non-root node in $\mathscr{F}_-$, counting across all compressions

Forest Slicing

- ***Claim:*** The cost of compressions crossing from $\mathscr{F}_-$ to $\mathscr{F}_+$ can be bounded by
  - the cost of some compressions done purely in $\mathscr{F}_+$ (the top parts of the compressions),
  - the total number of compressions from $\mathscr{F}_-$ to $\mathscr{F}_+$ (changing the parents of nodes in $\mathscr{F}_-$ whose parents are already in $\mathscr{F}_+$), and
  - the number of nodes in $\mathscr{F}_-$ whose parents are in $\mathscr{F}_-$ (each of which may get a parent in $\mathscr{F}_+$ for the first time at most once).

Recursively handle this when processing $\mathscr{F}_+$.

Happens once per compression from $\mathscr{F}_-$ to $\mathscr{F}_+$.

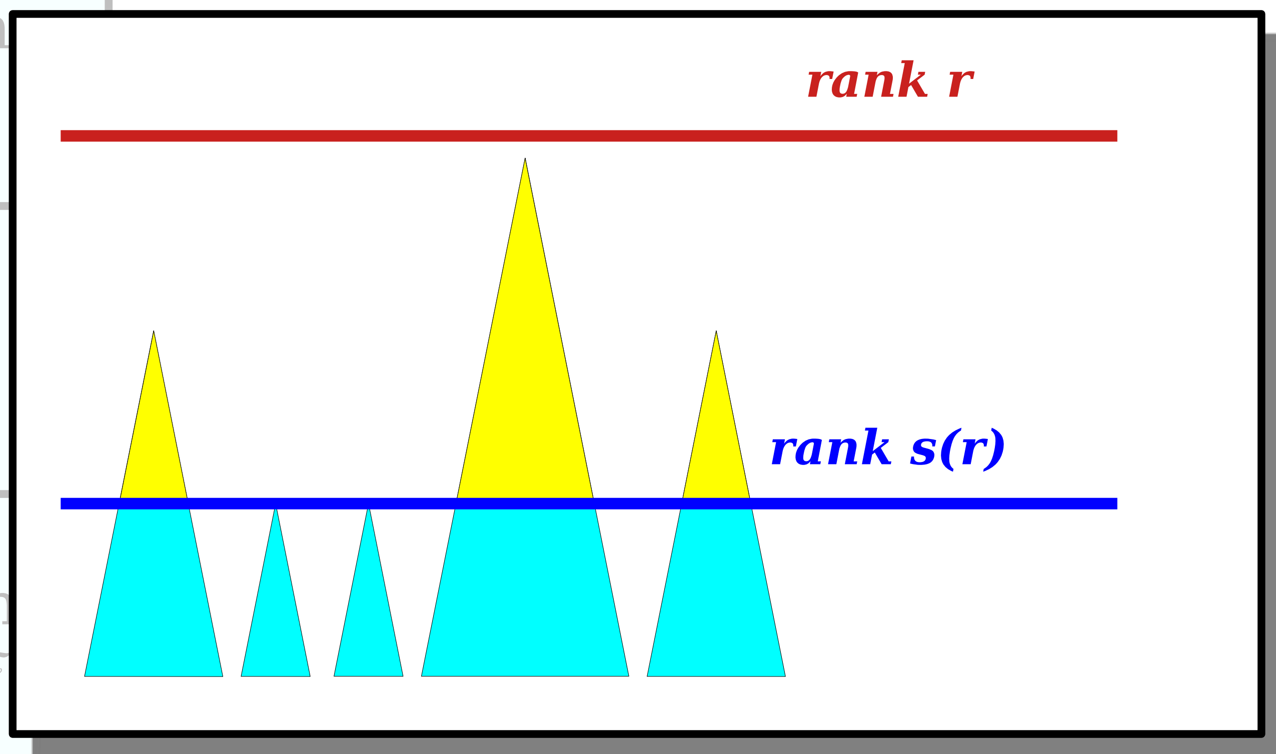
Happens once per non-root node in $\mathscr{F}_-$, counting across all compressions

Putting It All Together

- ***Claim:*** The cost of *all* the compressions performed in $\mathscr{F}$ is bounded by the following:
  - The cost of some compressions purely in $\mathscr{F}_-$.
  - The cost of some compressions purely in $\mathscr{F}_+$.
    - This includes compressions originally in $\mathscr{F}_+$, plus the "tops" of compressions from $\mathscr{F}_-$ to $\mathscr{F}_+$.
  - The number of compresses from $\mathscr{F}_-$ to $\mathscr{F}_+$.
    - This accounts for changing the parents of nodes in $\mathscr{F}_-$ whose parents are already in $\mathscr{F}_+$.
  - The number of nodes in $\mathscr{F}_-$ with parents in $\mathscr{F}_-$.
    - Each of these nodes may get a parent in $\mathscr{F}_+$ for the first time once.

$$C(m, n, r) \leq \ldots$$

Cost of compressions purely in $\mathscr{F}_+$.	$C(\,??\,, \boldsymbol{n_+}, \boldsymbol{r})$
Cost of compression purely in $\mathscr{F}_-$.	
The number of compresses from $\mathscr{F}_-$ to $\mathscr{F}_+$	
Number of nodes in $\mathscr{F}_-$ with parents in $\mathscr{F}$	



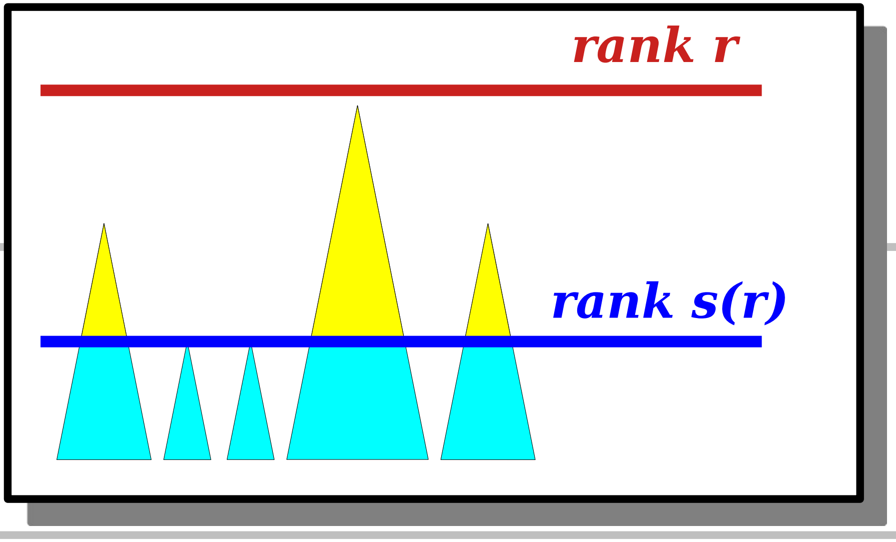
$$C(m, n, r) \leq \dots$$

Cost of compressions purely in $\mathscr{F}_+$.	$C(\boldsymbol{m_+}, \boldsymbol{n_+}, \boldsymbol{r})$
Cost of compressions purely in $\mathscr{F}_-$.	
The number of compresses from $\mathscr{F}_-$ to $\mathscr{F}_+$	
Number of nodes in $\mathscr{F}_-$ with parents in $\mathscr{F}_-$.	

The "tops" of all compressions running from $\mathscr{F}_-$ to $\mathscr{F}_+$ are handled in this bunch.

Let $\boldsymbol{m_+}$ be the number of compressions charged to $\mathscr{F}_+$, including both compressions purely within $\mathscr{F}_+$ and the "tops" of compressions crossing from $\mathscr{F}_-$ to $\mathscr{F}_+$.

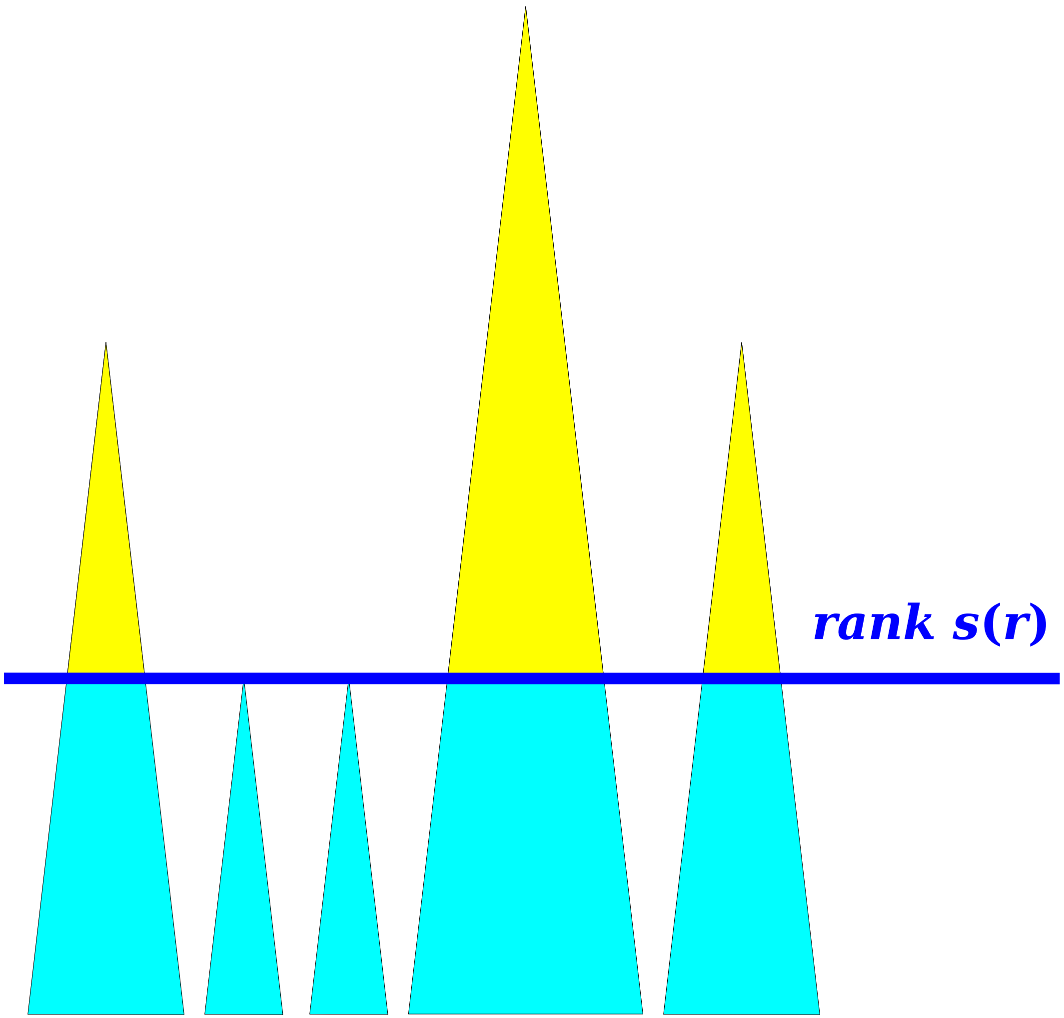
$$C(m, n, r) \leq \ldots$$

Cost of compressions purely in $\mathscr{F}_+$.	$C(m_+, n_+, r)$
Cost of compressions purely in $\mathscr{F}_-$.	$C(\boldsymbol{m - m_+}, \boldsymbol{n}, \boldsymbol{s(r)})$
The number of compresses from $\mathscr{F}_-$ to $\mathscr{F}_+$	
Number of nodes in $\mathscr{F}_-$ with parents in $\mathscr{F}_-$.	

$$C(m, n, r) \leq \ldots$$

Cost of compressions purely in $\mathscr{F}_+$.	$C(m_+, n_+, r)$
Cost of compressions purely in $\mathscr{F}_-$.	$C(m - m_+, n, s(r))$
The number of compresses from $\mathscr{F}_-$ to $\mathscr{F}_+$	m_+
Number of nodes in $\mathscr{F}_-$ with parents in $\mathscr{F}_-$.	

(Since this includes all compresses from $\mathscr{F}_-$ to $\mathscr{F}_+$).



Every node in $\mathscr{F}_+$ has rank $s(r) + 1$ or greater.

Each rank-k node has children of ranks $0, 1, 2, \ldots, k - 1$.

So every node in $\mathscr{F}_+$ has at least $s(r) + 1$ children, and they're all in in $\mathscr{F}_-$.

There are n total nodes, and n_+ of them are in $\mathscr{F}_+$.

Nodes in $\mathscr{F}_-$: $\boldsymbol{n - n_+}$.

Nodes in $\mathscr{F}_-$ whose parents are in $\mathscr{F}_+$: $\boldsymbol{n_+ \cdot (s(r) + 1))}$

Nodes in $\mathscr{F}_-$ with parents in $\mathscr{F}_-$: $\boldsymbol{n - n_+ \cdot (s(r) + 2)}$.

Number of nodes in $\mathscr{F}_-$ with parents in $\mathscr{F}_-$.	$n - n_+ \cdot (s(r) + 2)$

$$C(m, n, r) \leq \ldots$$

Cost of compressions purely in $\mathscr{F}_+$.	$C(m_+, n_+, r)$
Cost of compressions purely in $\mathscr{F}_-$.	$C(m - m_+, n\ , s(r))$
The number of compresses from $\mathscr{F}_-$ to $\mathscr{F}_+$	m_+
Number of nodes in $\mathscr{F}_-$ with parents in $\mathscr{F}_-$.	$n - n_+ \cdot s(r)$

The Recurrence

- Putting it all together:

$$\begin{aligned} \boldsymbol{C(m, n, r)} \ \leq\ & \boldsymbol{C(m - m_+, n, s(r)) +} \\ & \boldsymbol{C(m_+, n_+, r) +} \\ & \boldsymbol{m_+ + n - n_+ \cdot s(r).} \end{aligned}$$

- Now, “all” we need to do is solve this.
- Don’t panic! This is indeed tricky, but it’s not as bad as it looks.

The Recurrence

$$
\begin{aligned}
C(m, n, r) \le\ & C(m - m_+, n, s(r)) + \\
& C(m_+, n_+, r) + \\
& m_+ + n - n_+ \cdot s(r).
\end{aligned}
$$

- ***Recall:*** $C(m, n, r) \le m + n \cdot r / 2$.

The Recurrence

$$C(m, n, r) \leq C(m - m_+, n, s(r)) + m_+ + n_+ \cdot r / 2 + m_+ + n - n_+ \cdot s(r).$$

- ***Recall:*** $C(m, n, r) \leq m + n \cdot r / 2$.

The Recurrence

$$C(m, n, r) \le C(m - m_+, n, s(r)) + 2m_+ + n + n_+ \cdot (r / 2 - s(r)).$$

- ***Recall:*** $C(m, n, r) \le m + n \cdot r / 2$.

The Recurrence

$$C(m, n, r) \le C(m - m_+, n, s(r)) + 2m_+ + n + n_+ \cdot (r / 2 - s(r)).$$

- ***Clever Decision:*** Set $s(r) = r / 2$.

The Recurrence

$$C(m, n, r) \leq C(m - m_+, n, r / 2) + 2m_+ + n.$$

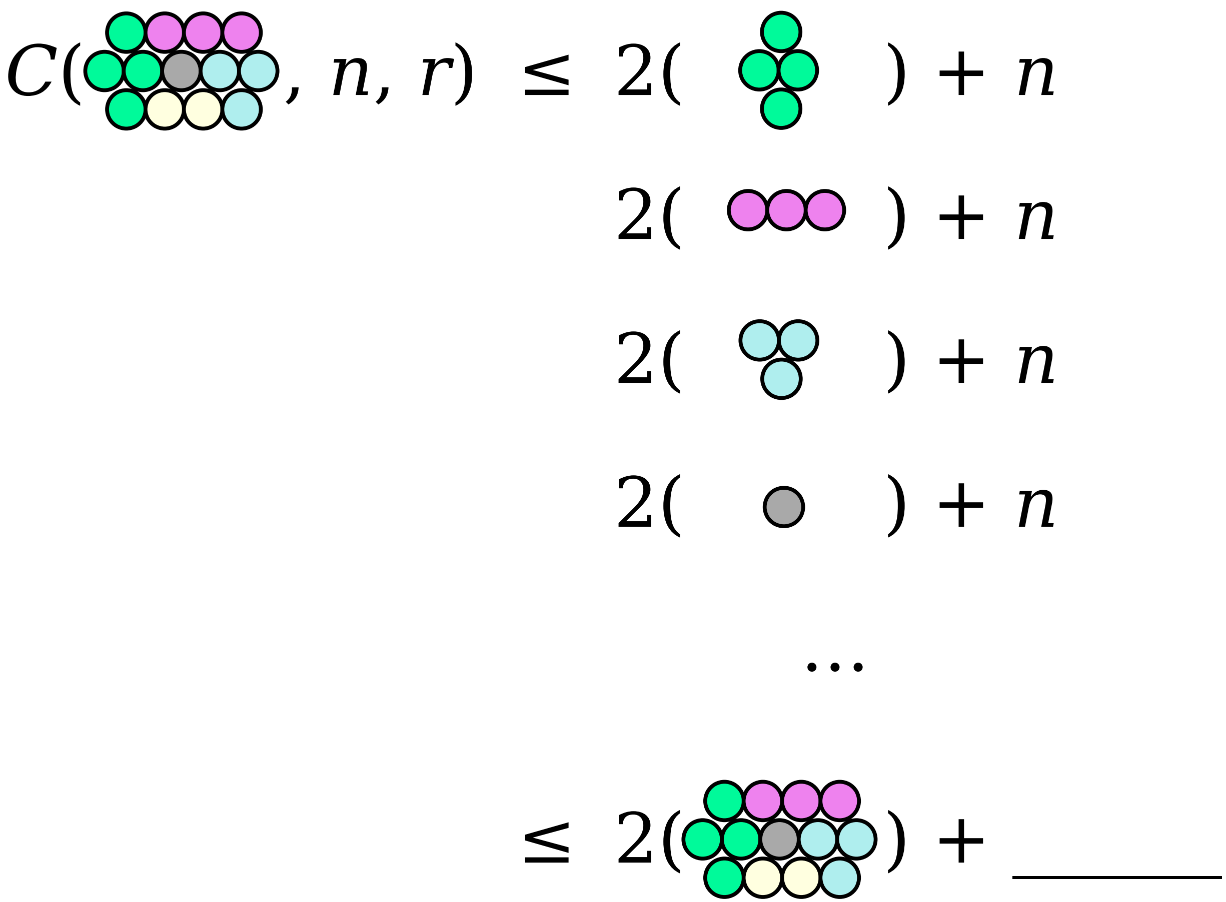
- ***Clever Decision:*** Set $s(r) = r / 2$.

$$C(m, n, r) \leq C(m - m_{+}, n, r / 2) + 2m_{+} + n$$

Starting with m…

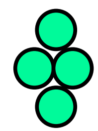
…take some of m away…

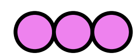
… and add it into the total.

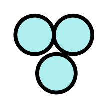


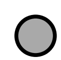
$$C(m, n, r) \le C(m - m_+, n, r / 2) + 2m_+ + n$$

How many layers can this recursion have?

$C(m, n, r) \le 2($  $) + n$

$2($  $) + n$

$2($  $) + n$

$2($  $) + n$

…

$$\le 2m + n \lg r$$

Where We Are

- We've just proven that

$$\boldsymbol{C(m, n, r) \leq 2m + n \lg r}.$$

- The maximum rank in an n-node forest is $r = O(\lg n)$.
- This gives a bound of $\mathbf{O(\boldsymbol{m} + \boldsymbol{n} \log \log \boldsymbol{n})}$ for any series of operations.
- That's a lot better than the $O(m \log n)$ we started with – and it's just due to better accounting, rather than a fundamental reenvisioning of the data structure.
- Is this a tight bound, or can we do better?

The Recurrence

$$C(m, n, r) \leq C(m - m_+, n, s(r)) + C(m_+, n_+, r) + m_+ + n - n_+ \cdot s(r).$$

- ~~***Recall:*** $C(m, n, r) \leq m + n \cdot r / 2.$~~
- ***Recall:*** $C(m, n, r) \leq 2m + n \lg r.$

The Recurrence

$$
\begin{aligned}
C(m, n, r) \le\ & C(m - m_+, n, s(r)) + \\
& 2m_+ + n_+ \cdot \lg r + \\
& m_+ + n - n_+ \cdot s(r).
\end{aligned}
$$

- ~~***Recall:*** $C(m, n, r) \le m + n \cdot r / 2.$~~
- ***Recall:*** $C(m, n, r) \le 2m + n \lg r.$

The Recurrence

$$C(m, n, r) \le C(m - m_+, n, s(r)) + 3m_+ + n + n_+ \cdot (\lg r - s(r)).$$

- ~~***Recall:*** $C(m, n, r) \le m + n \cdot r / 2$.~~
- ***Recall:*** $C(m, n, r) \le 2m + n \lg r$.

The Recurrence

$$
\begin{aligned}
C(m, n, r) \leq\ & C(m - m_+, n, s(r)) + \\
& 3m_+ + n + \\
& n_+ \cdot (\lg r - s(r)).
\end{aligned}
$$

- ***Clever Decision:*** Set $s(r) = \lg r$.

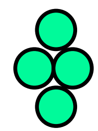
The Recurrence

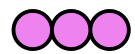
$$C(m, n, r) \le C(m - m_+, n, \lg r) + 3m_+ + n$$

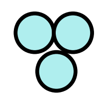
- ***Clever Decision:*** Set $s(r) = \lg r$.

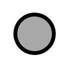
$$C(m, n, r) \le C(m - m_+, n, \lg r) + 3m_+ + n$$

How many layers can this recursion have?

$\boldsymbol{C(m, n, r)}$ $\le$ $3($  $) + n$

$3($  $) + n$

$3($  $) + n$

$3($  $) + n$

...

$$\boldsymbol{\le 3m + n \lg^* r}$$

Where We Are

- We've just proven that

$$\boldsymbol{C(m, n, r) \leq 3m + n \lg^* r}.$$

- Since $r = O(\log n)$ this gives a bound of **$O(m + n \log^* n)$** for any series of operations.
- That's a ***substantial*** improvement over our previous bound – and all we did was feed the analysis back into itself!
- ***Can we do better?***

Notice Something?

If we start with this bound on $C(m, n, \mathrm{r})$…	… then we get this stronger bound on $C(m, n, r)$:
$m + n \cdot r / 2$	$2m + n \lg r$
$2m + n \lg r$	$3m + n \lg^* r$
$km + n\, f(r)$	$(k+1)m + n\, f^*(r)$

The Recurrence

$$\begin{aligned} C(m, n, r) \leq\ & C(m - m_+, n, s(r)) + \\ & C(m_+, n_+, r) + \\ & m_+ + n - n_+ \cdot s(r). \end{aligned}$$

- ***Assume:*** $C(m, n, r) \leq km + n \cdot f(r)$

The Recurrence

$$\begin{aligned} C(m, n, r) \le\ & C(m - m_+, n, s(r)) + \\ & (k+1)m_+ + n + \\ & n_+ \cdot (f(r) - s(r)). \end{aligned}$$

- ***Assume:*** $C(m, n, r) \le km + n \cdot f(r)$

The Recurrence

$$C(m, n, r) \le C(m - m_+, n, s(r)) + (k+1)m_+ + n + n_+ \cdot (f(r) - s(r)).$$

- ***Clever Idea:*** Set $s(r) = f(r)$.

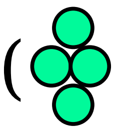
The Recurrence

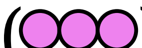
$$C(m, n, r) \le C(m - m_+, n, f(r)) + (k+1)m_+ + n.$$

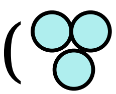
- ***Clever Idea:*** Set $s(r) = f(r)$.

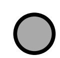
$$C(m, n, r) \le C(m - m_+, n, f(r)) + (k+1)m_+ + n$$

How many layers can this recursion have?

$\boldsymbol{C(m, n, r)} \le (k+1)($  $) + n$

$(k+1)($  $) + n$

$(k+1)($  $) + n$

$(k+1)($  $) + n$

...

$$\boldsymbol{\le (k+1)m + nf^*(r)}$$

Interpreting This Result

- We now have a family of bounds on the cost of operations on a disjoint-set forest:

$$m + n \cdot ({}^{r}/_{2})$$

$$2m + n \lg r$$

$$3m + n \lg^{*} r$$

$$4m + n \lg^{**} r$$

$$5m + n \lg^{***} r$$

$$\ldots$$

- Which of these is the “best” bound?

Interpreting This Result

- For now, focus on these bounds:

$$2m + n \lg r$$

$$3m + n \lg^* r$$

$$4m + n \lg^{**} r$$

$$5m + n \lg^{***} r$$

$$6m + n \lg^{****} r$$

- More generally, we have bounds of the form

$$(k + 2)m + n \lg^{*(k)} r.$$

- There's some point at which making k larger makes that first term larger without decreasing the second term.
- What is it?

The Ackermann Inverse

- The ***Ackermann inverse function***, denoted $\boldsymbol{\alpha(z)}$, is defined as follows:

$$\boldsymbol{\alpha(z) = \min\{\ k \in \mathbb{N} \mid \log^{*(k)} z \leq 1\ \}}$$

- Intuitively, this counts how many times you have to put stars on log***$^{\dots}$*** z before it drops to 1.
- This function grows more slowly than anything in the iterated logarithm family – and that should give you a sense of just how slowly this function grows!
- ***Worthwhile Activity:*** find the smallest natural numbers where α produces 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10.

The Ackermann Inverse

- We have a bound of

$$\boldsymbol{(k + 2)m + n \log^{*(k)} r.}$$

- Picking $k = \alpha(r) = \alpha(\log n)$, and the bound on the cost of any series of m operations is.

$$\boldsymbol{O(m\alpha(\log n) + n)}.$$

- This is *essentially* "O(m + n)," because that α term is a constant for any input that could ever be fed in with the resources we know about in the universe. But technically speaking it's superlinear.

A Tighter Analysis

- By being a bit more clever with the analysis, we can tighten the bound as follows.
- Define $\boldsymbol{\alpha(m, n)}$ as

  $\boldsymbol{\alpha(m, n) = \min\{\, k \in \mathbb{N} \mid \log^{*(k)} (m / n) \leq \log n \,\}}$.
- Then the cost of m operations on an n-element forest can be shown to be $O(m\alpha(m, n))$, a slight improvement over what we just did here.

Major Ideas for Today

- Iterated functions generalize the idea of “how many times can you divide by two before you run out of things?”
- Iterated logarithms are a family of very slowly-growing functions, each of which grows more slowly than the previous one.
- The Ackermann inverse function grows slower than any number of iterated logarithms and essentially count what level of iteration is needed to clear a number.

Next Time

- ***Euler Tour Trees***
  - Fully dynamic connectivity in forests.
- ***Augmented Dynamic Trees***
  - Figuring out information about connected components in sublinear time.