

# Splay Trees

Recap from Last Time

<b><i>Property</i></b>	<b><i>Description</i></b>	<b><i>Met by</i></b>
<i>Balance</i>	Lookups take time $O(\log n)$ .	Traditional balanced BST
<i>Entropy</i>	Lookups take expected time $O(1 + H)$ .	Weight-equalized trees
<i>Dynamic Finger</i>	Lookups take $O(\log \Delta)$ . $\Delta$ measures distance.	Level-linked BST with finger
<i>Working Set</i>	Lookups take $O(\log t)$ , $t$ measures recency.	Iacono's structure

What does an “optimal” binary search tree look like?

Consider a discrete probability distribution with elements  $x_1, \dots, x_n$ , where element  $x_i$  has access probability  $p_i$ .

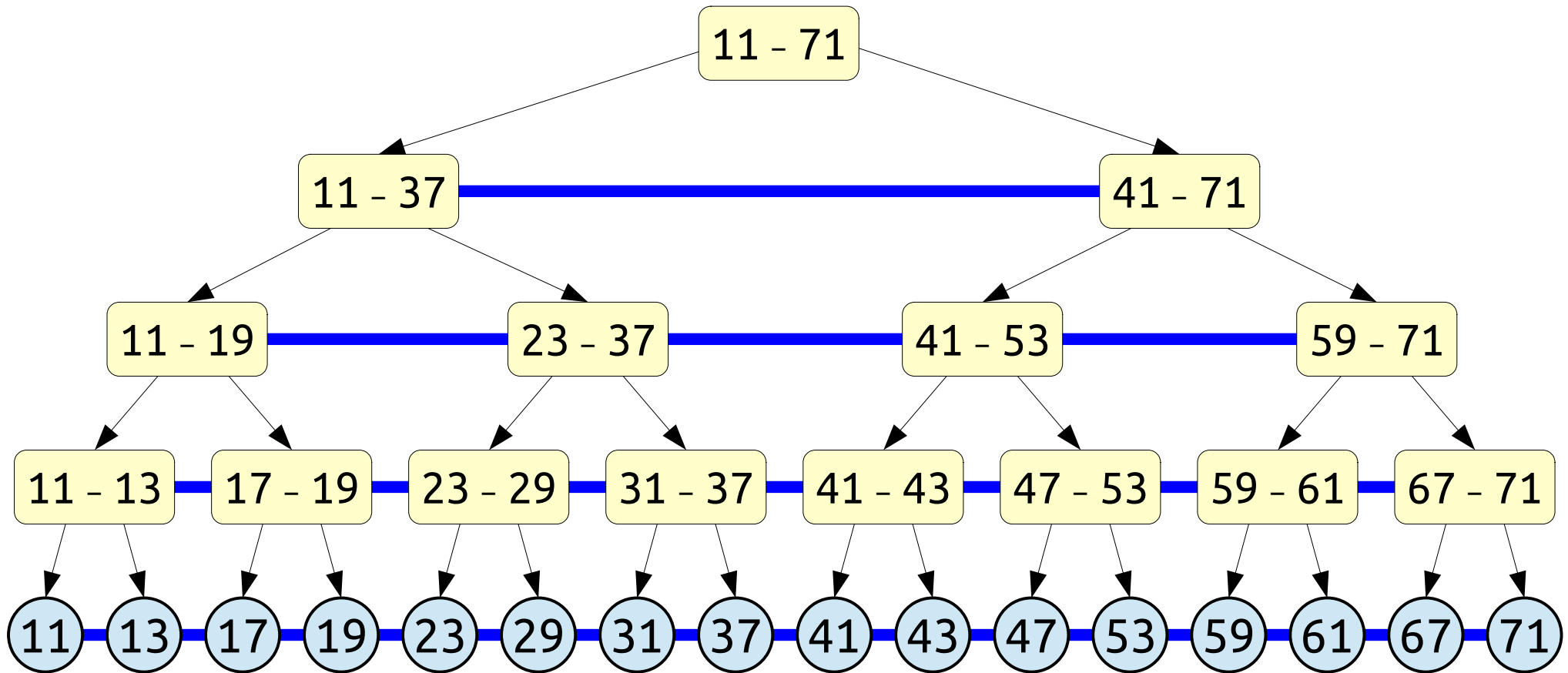
The **Shannon entropy** of this probability distribution, denoted  $H_p$  (or just  $H$ , where  $p$  is implicit) is the quantity

$$H_p = \sum_{i=1}^n p_i \lg \frac{1}{p_i}.$$

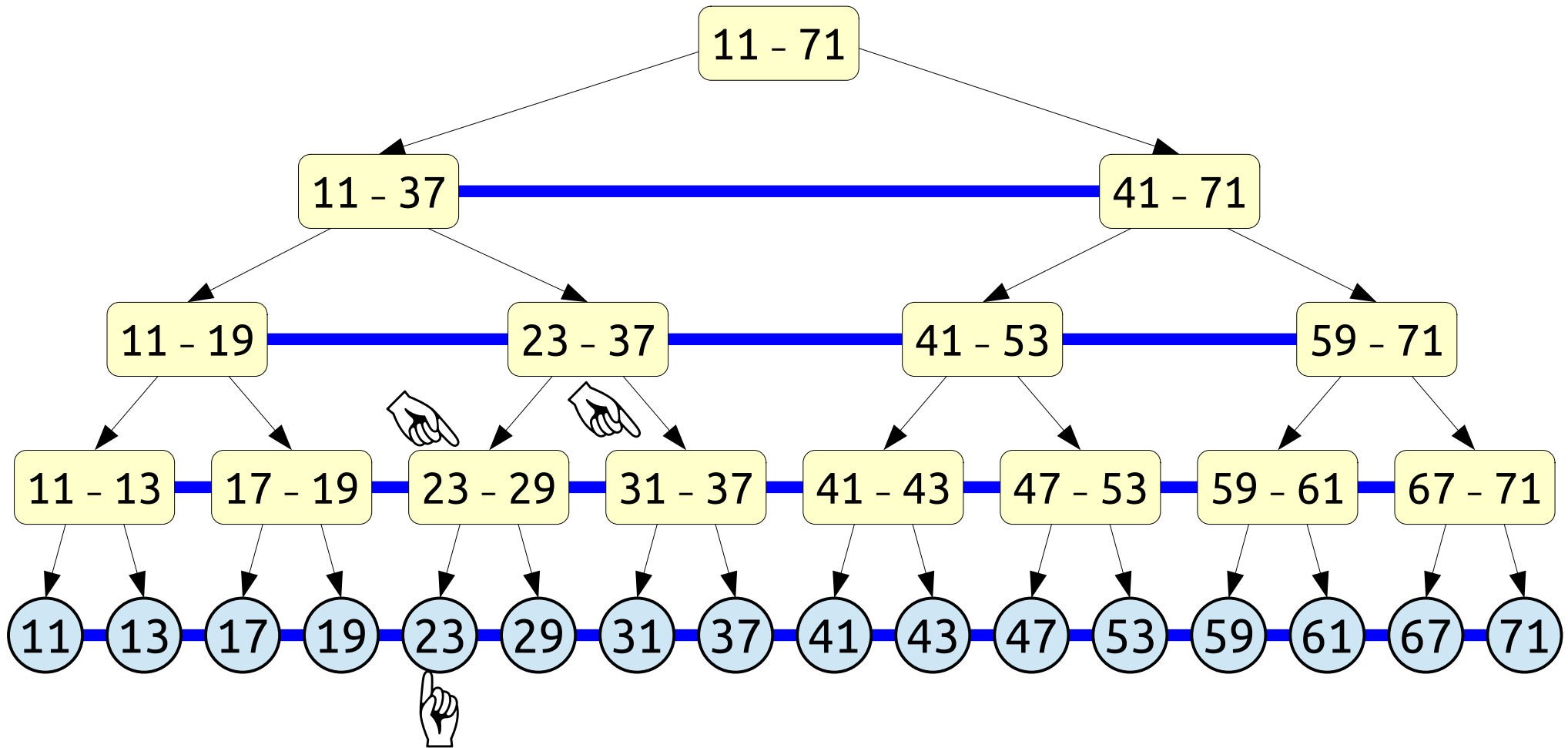
**Theorem:** The expected cost of a lookup in *any* BST with keys  $x_1, \dots, x_n$  and access probabilities  $p_1, \dots, p_n$  is  $\Omega(1 + H)$ .

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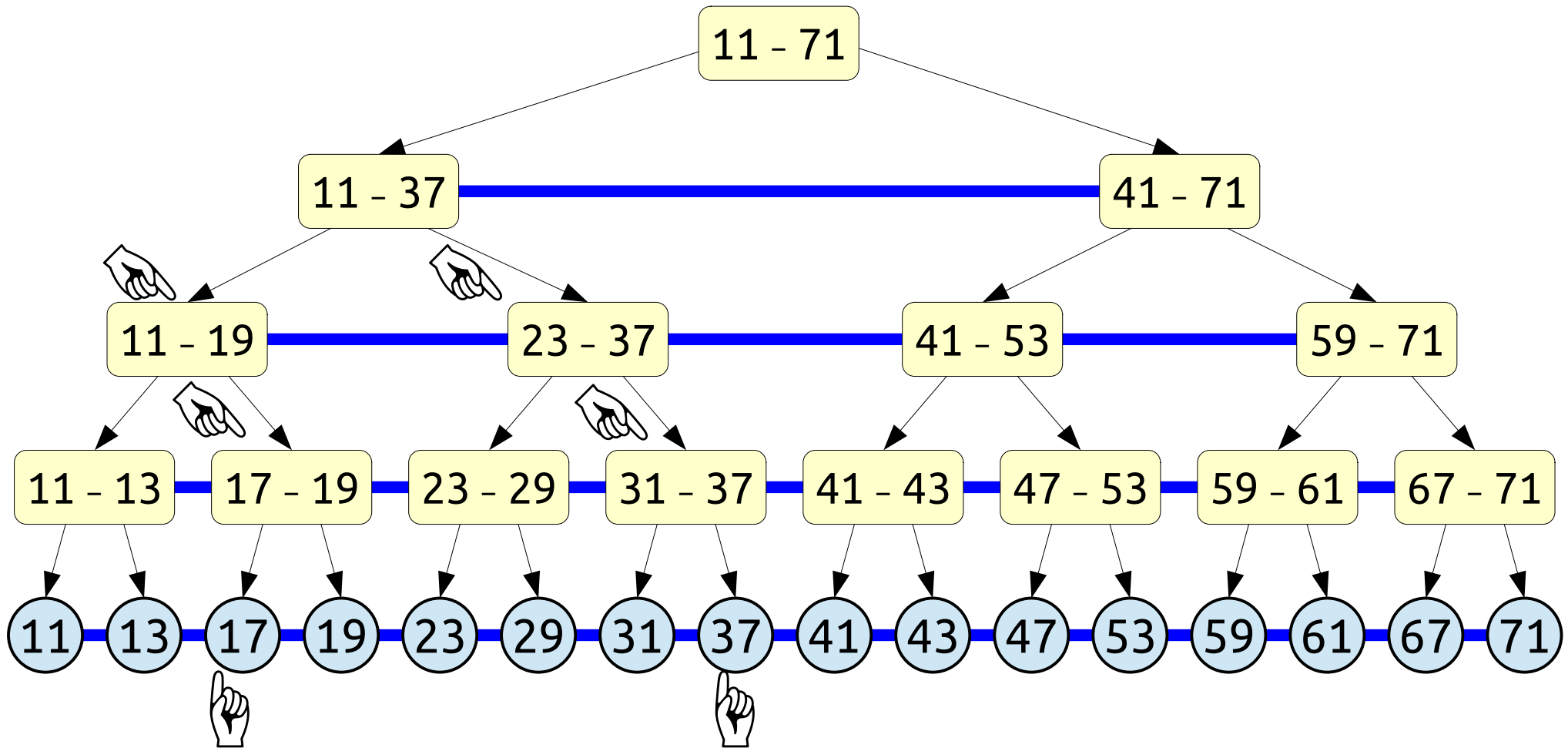
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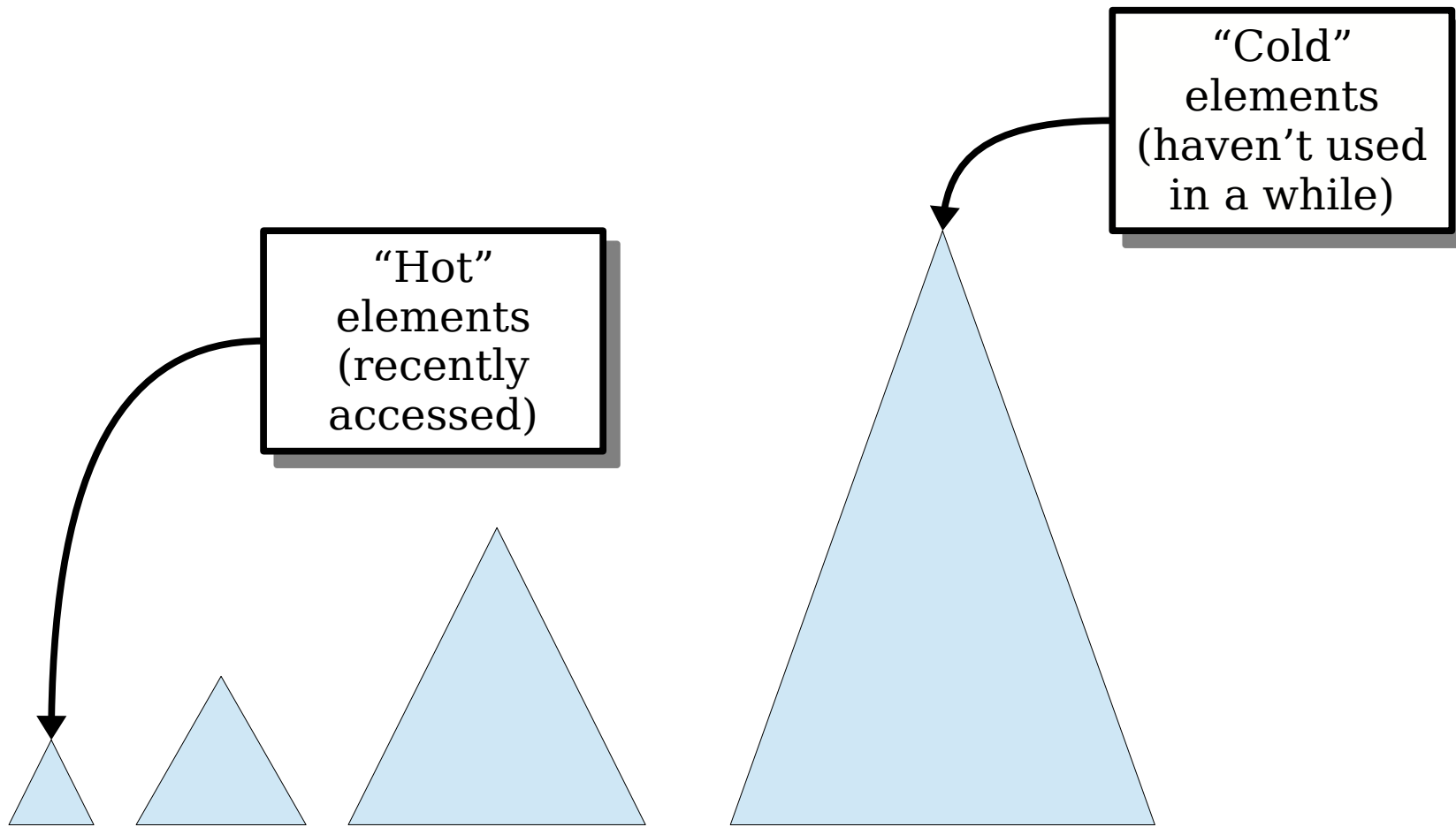
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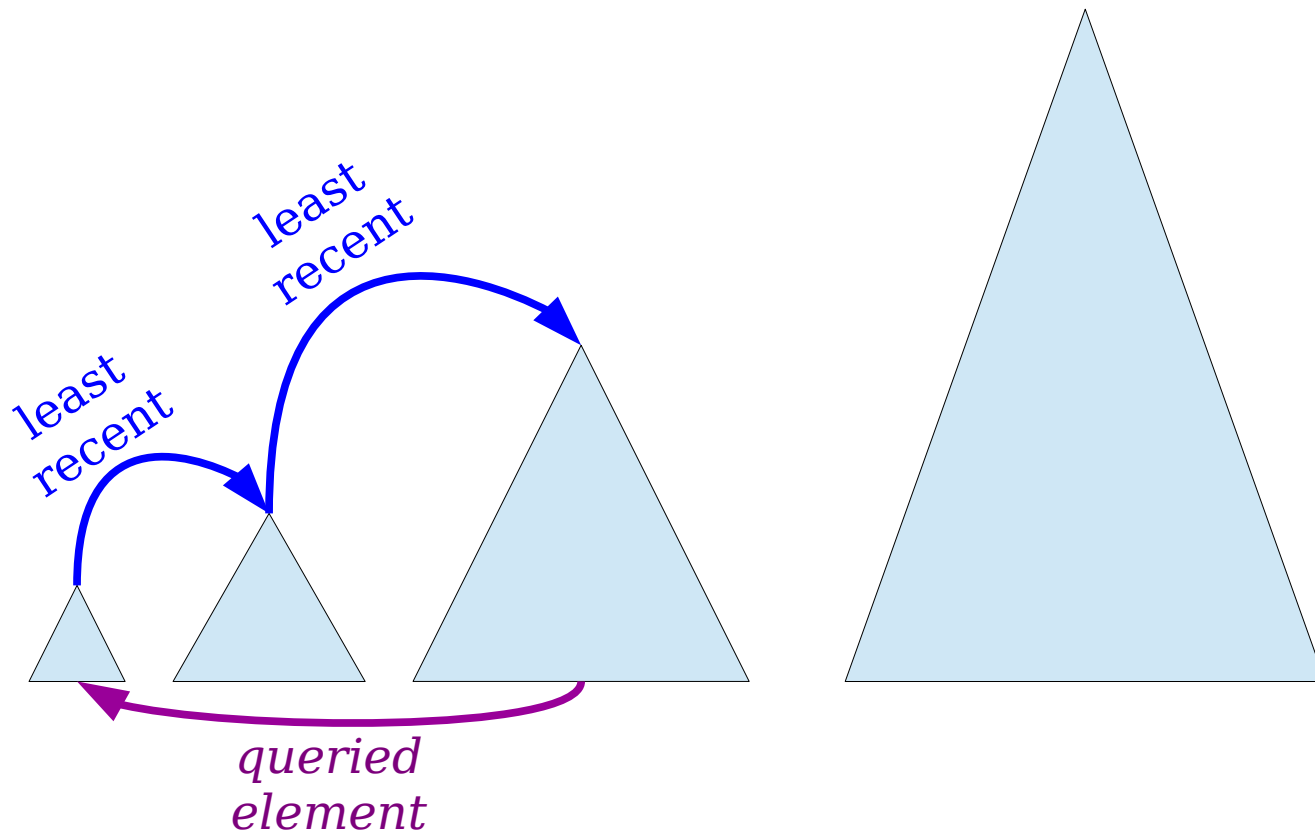
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Is there a single BST with all of these properties?

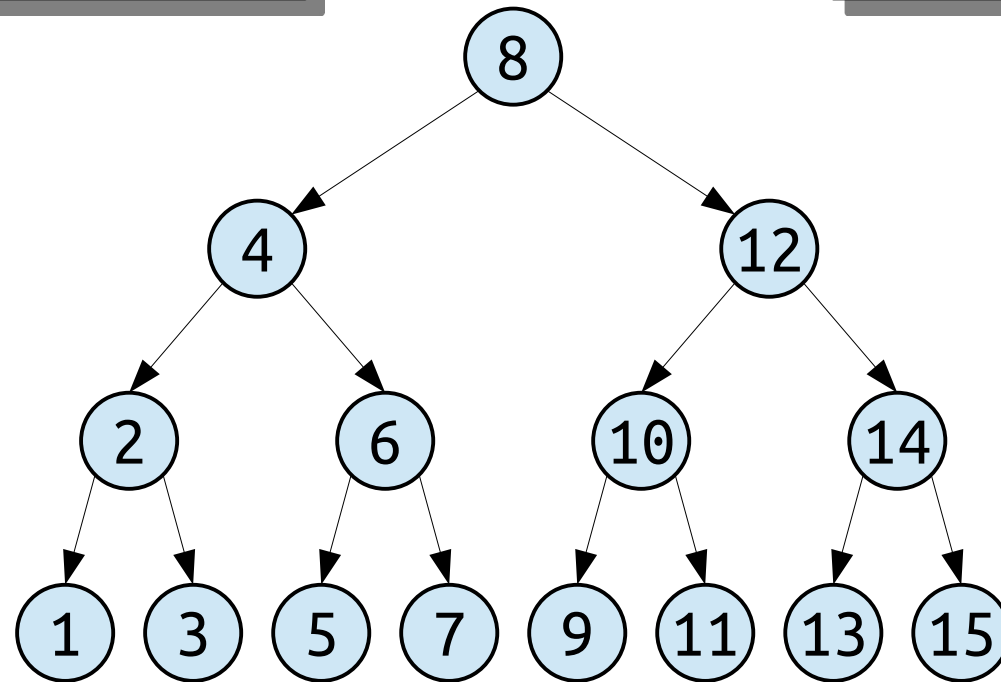
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***Yes!***

New Stuff!

**Idea 1:** Get the working set property by choosing a clever BST shape.

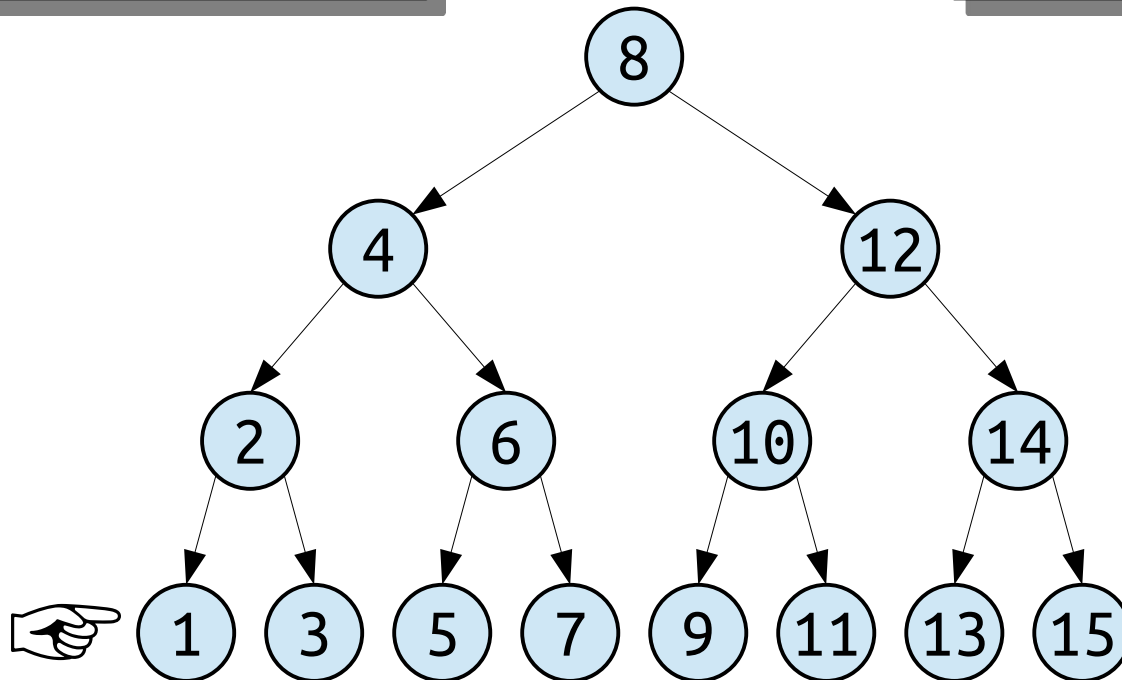
**Problem:** We can always pick a set of hot elements deep in the tree.



How do we build a BST with the working set property?

**Idea 2:** Get the working set property by adding a finger into our BST.

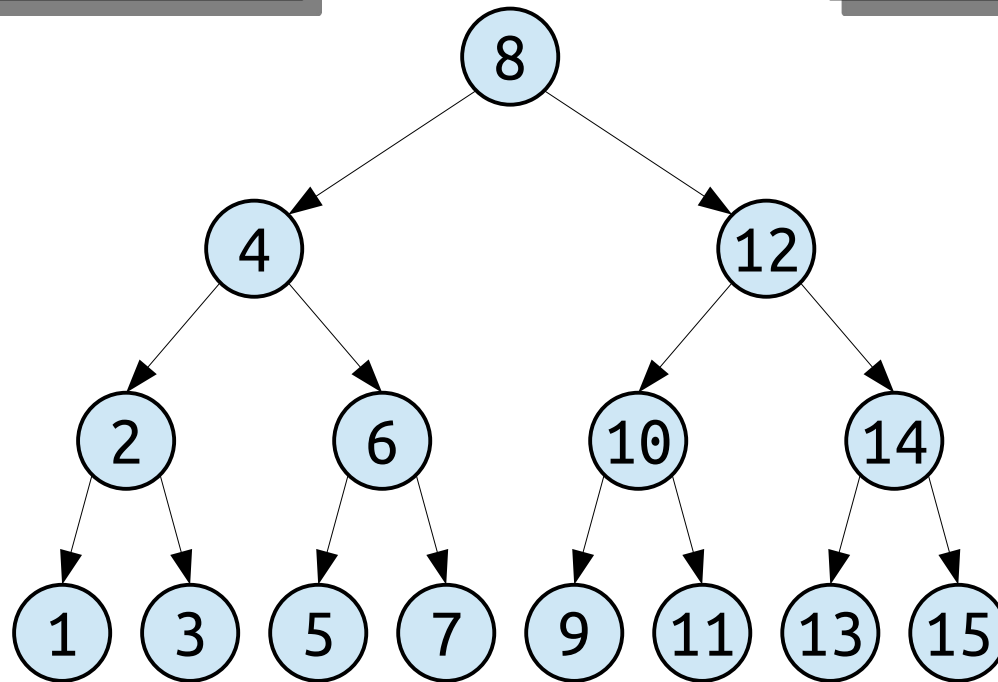
**Problem:** What if those keys aren't near each other in key space?



How do we build a BST with the working set property?

**Idea 3:** Get the working set property by moving nodes around the BST.

**Strategy:** After querying a node, rotate it up to the root of the tree.

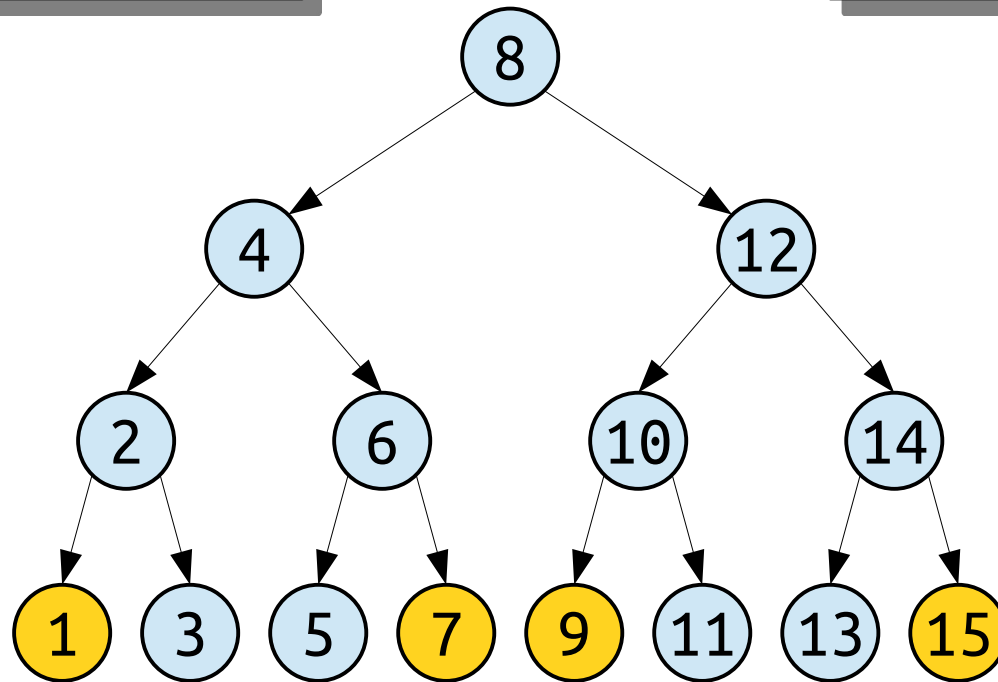


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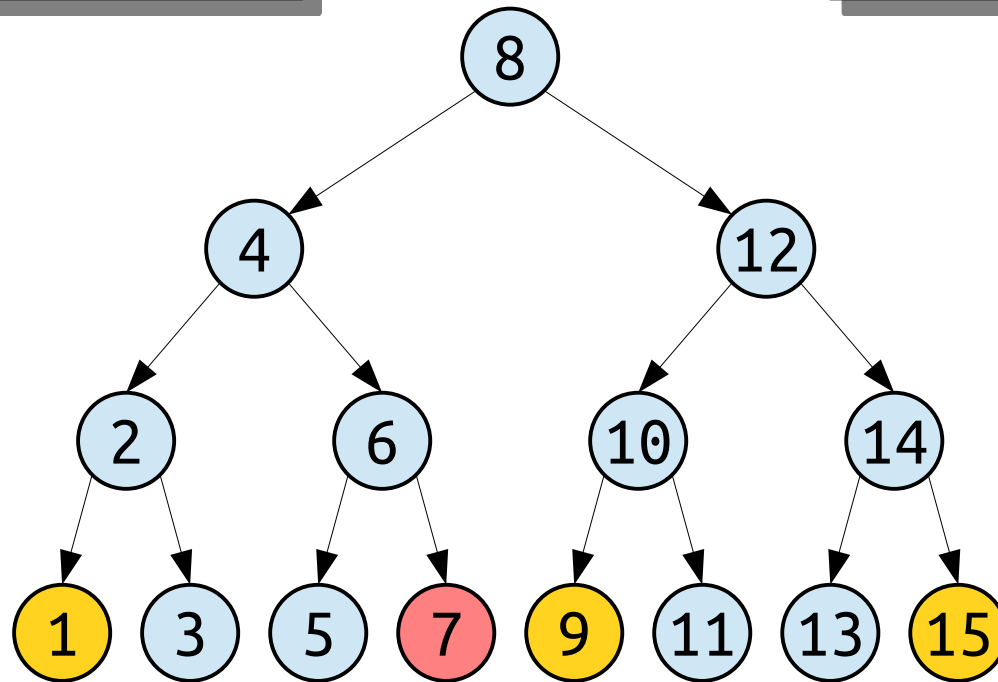


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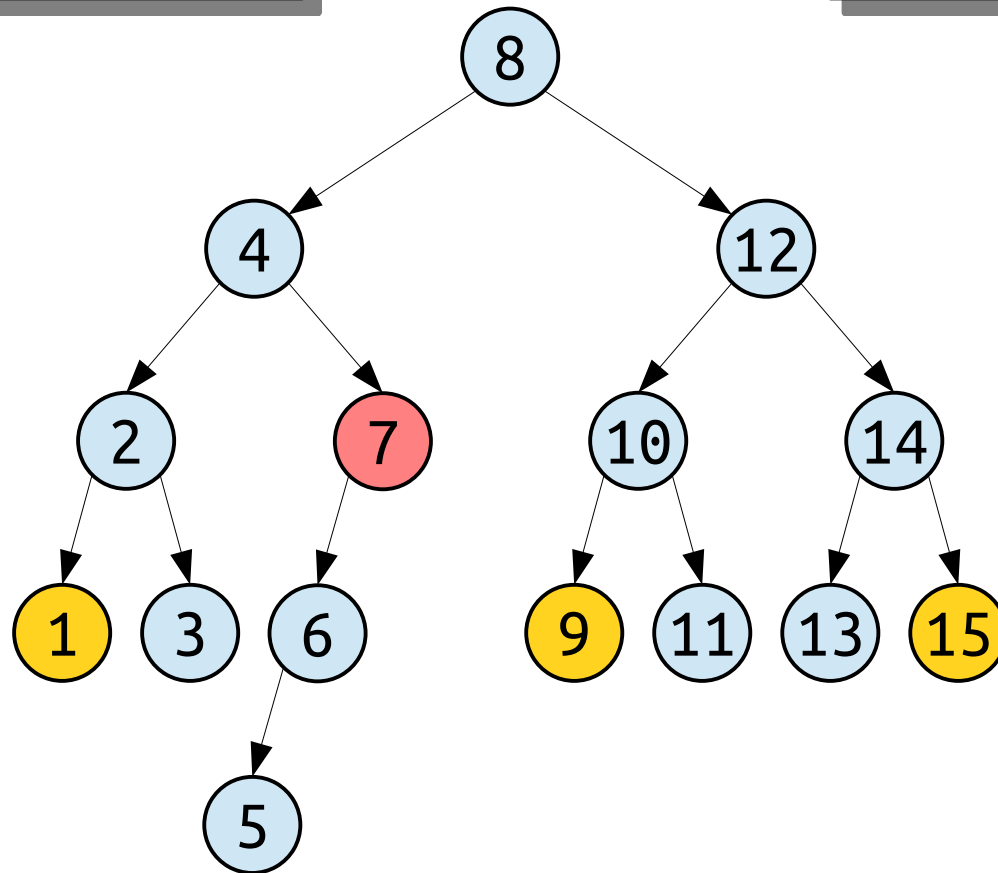
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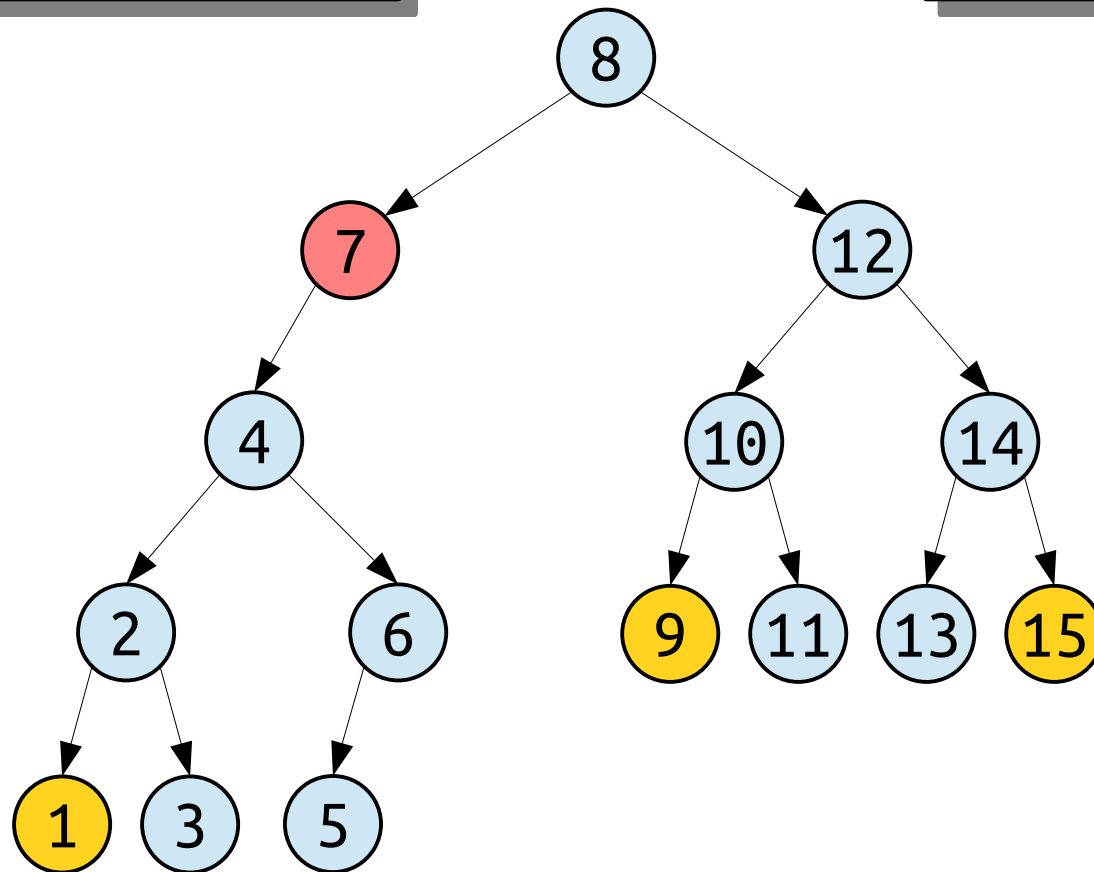
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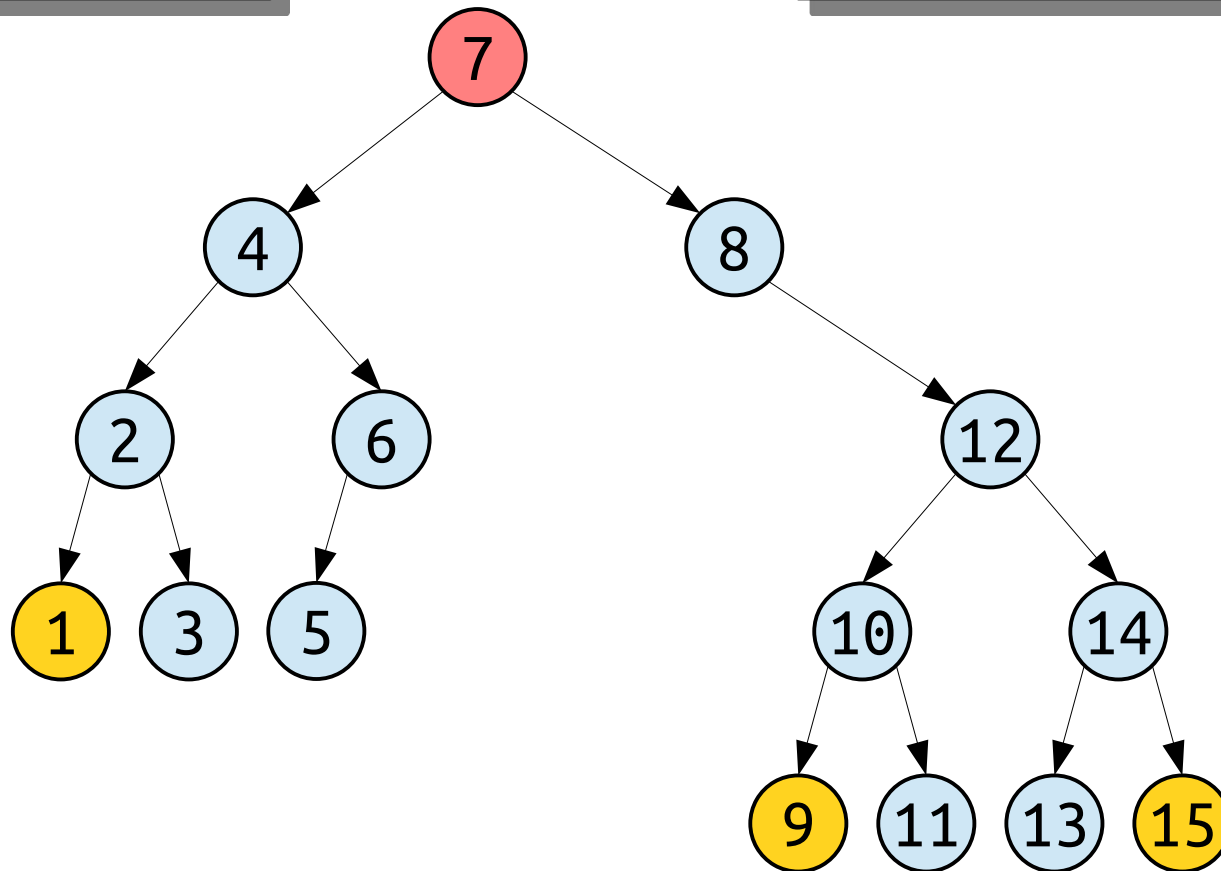
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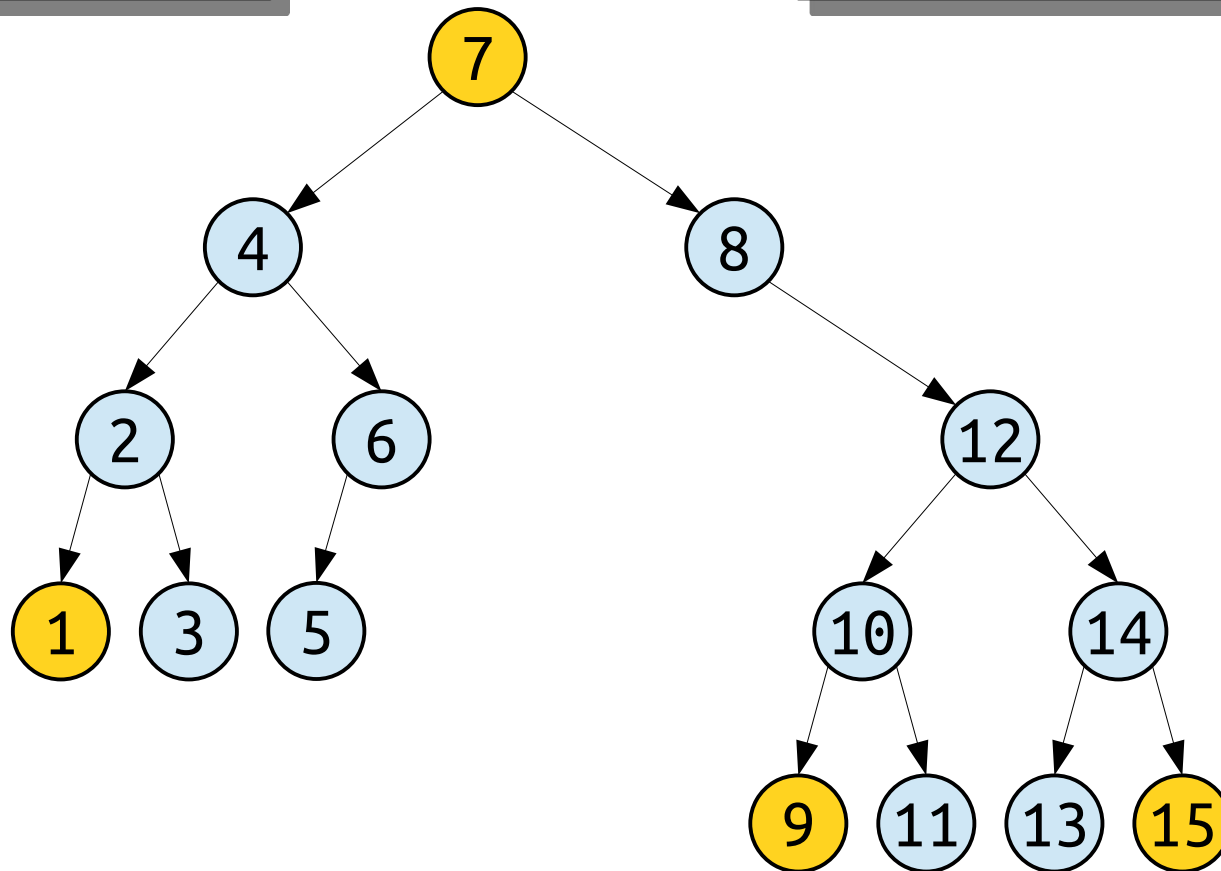
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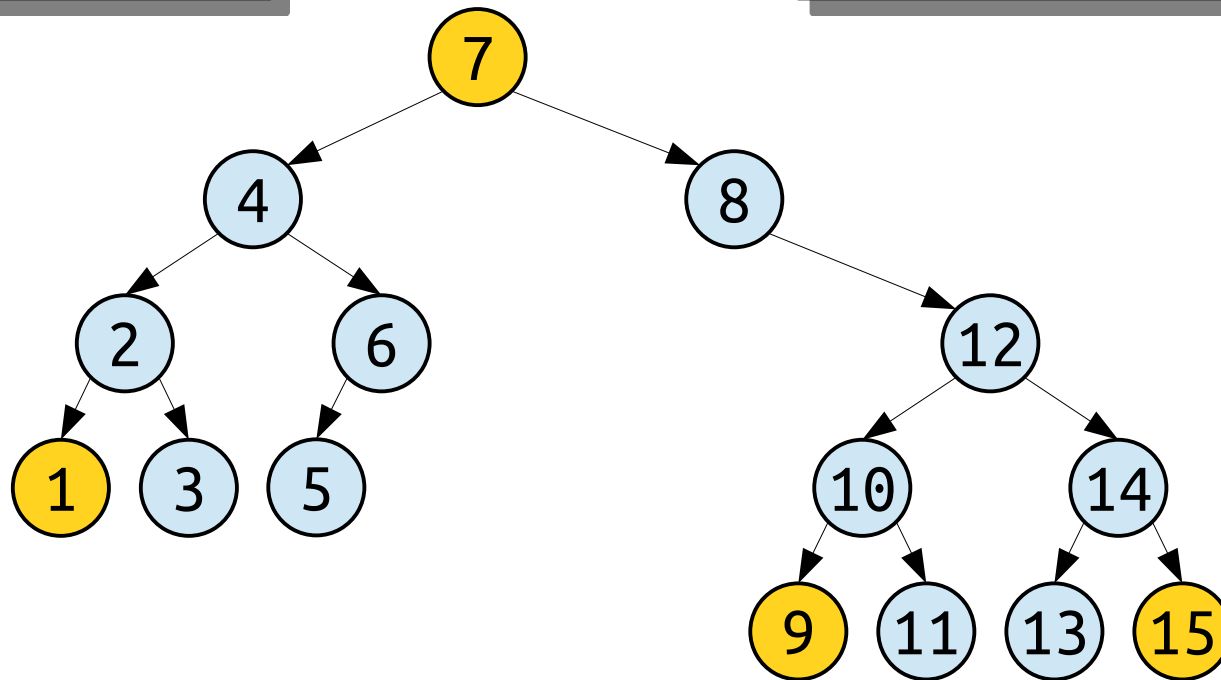
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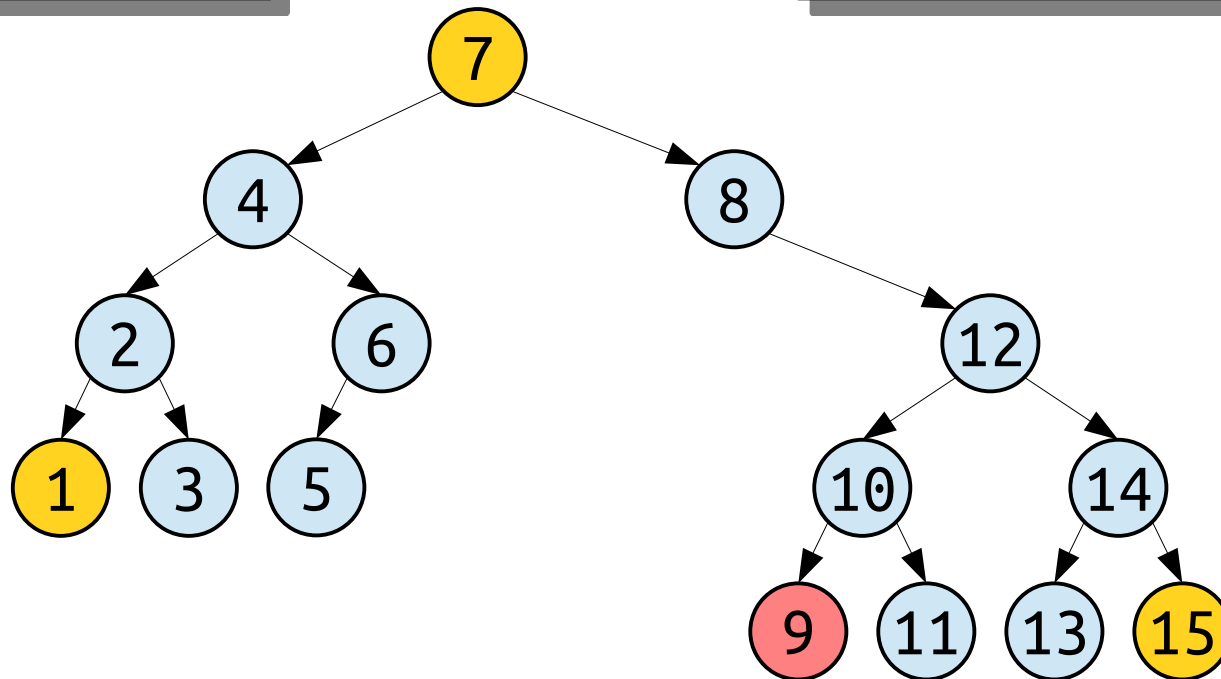
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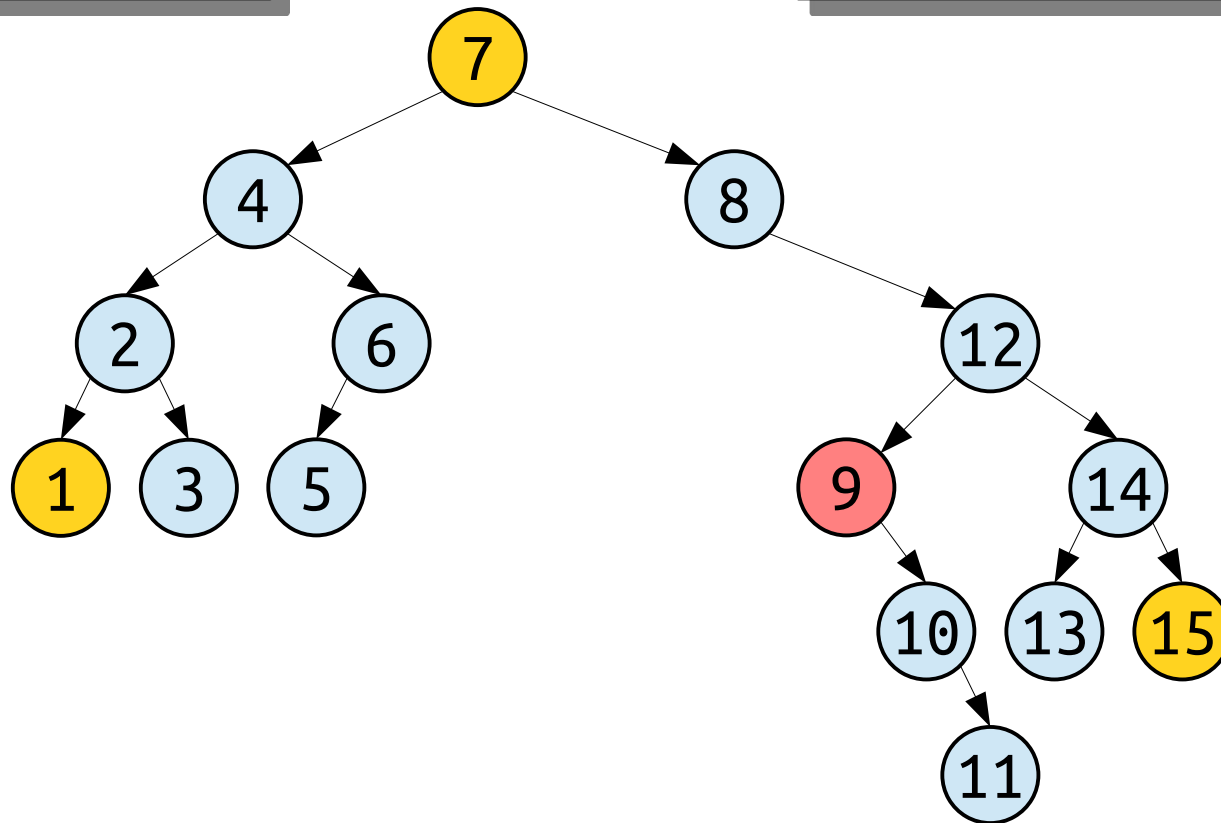
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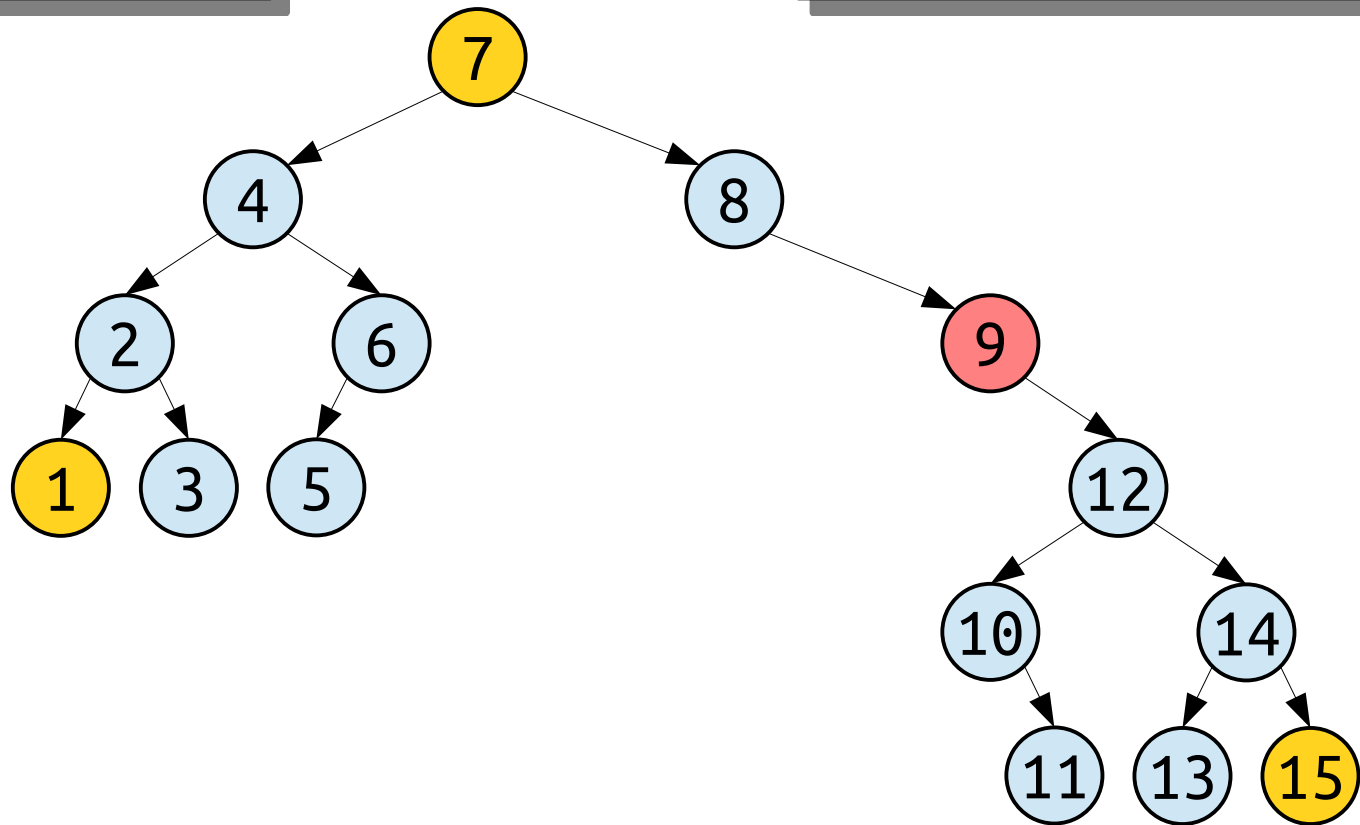


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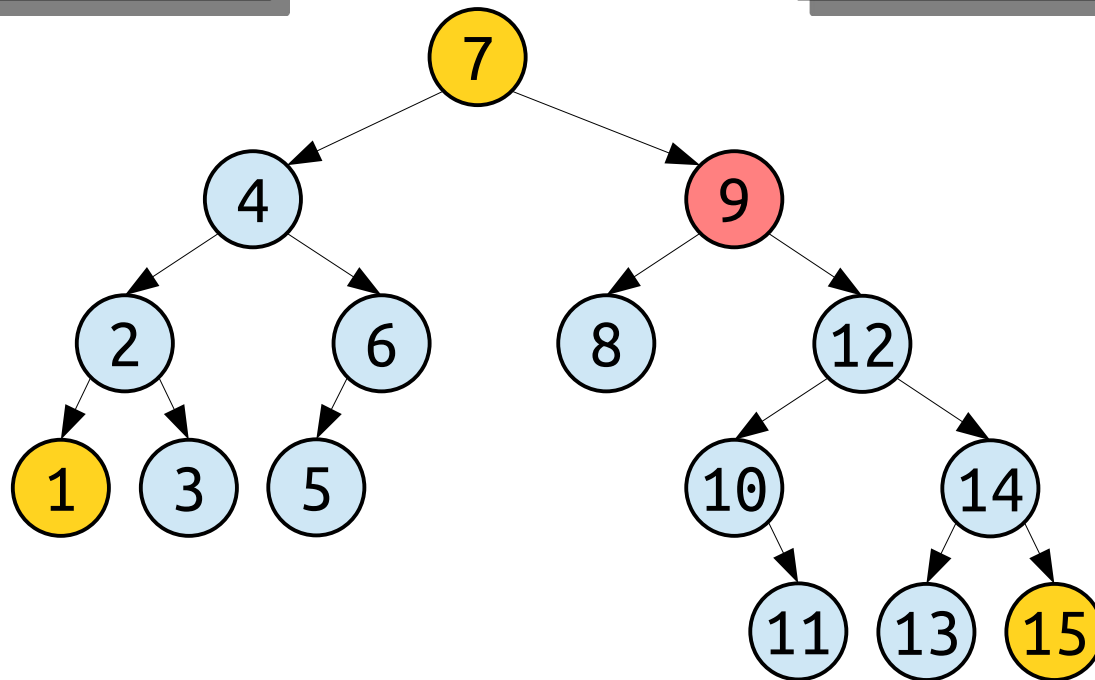
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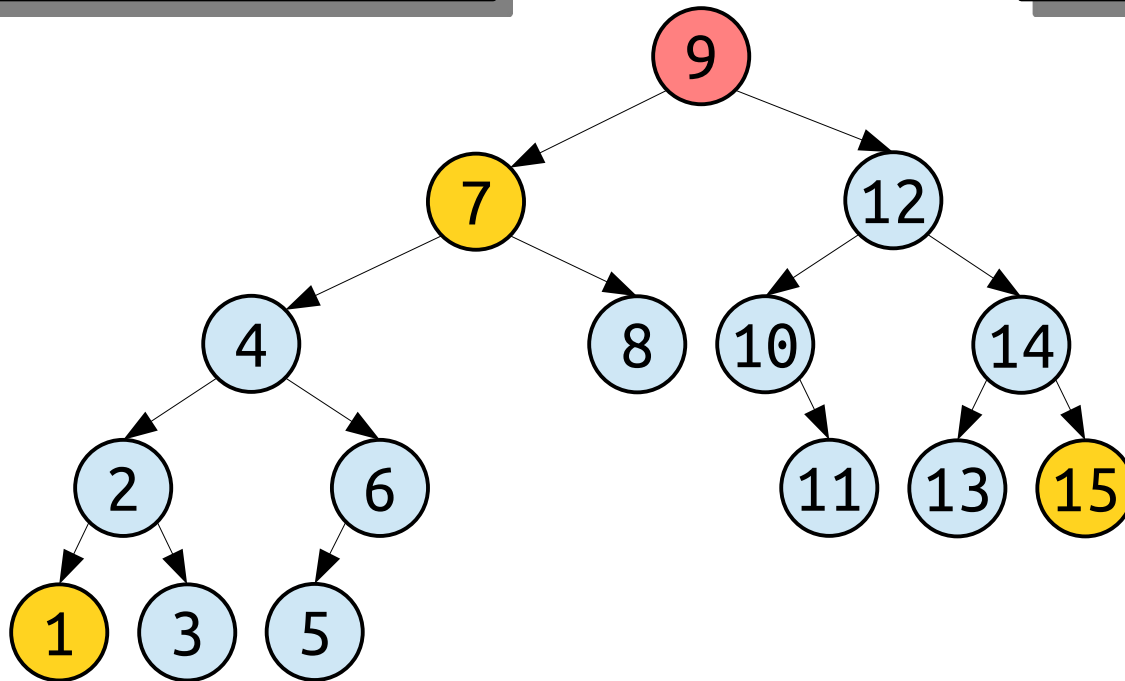
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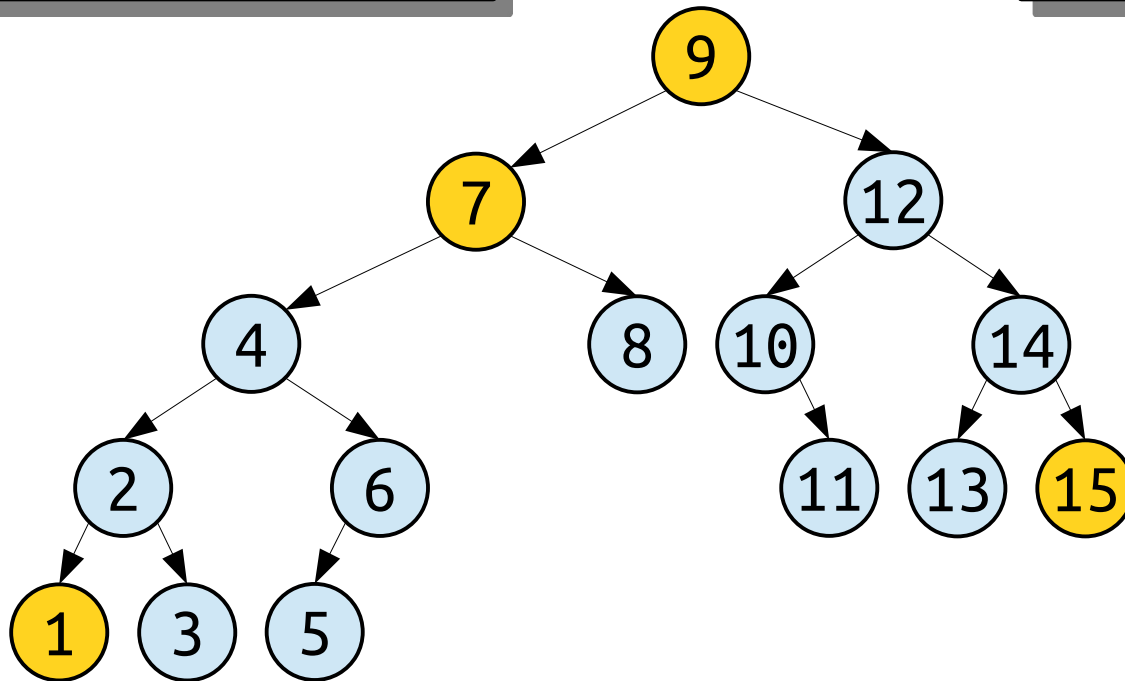
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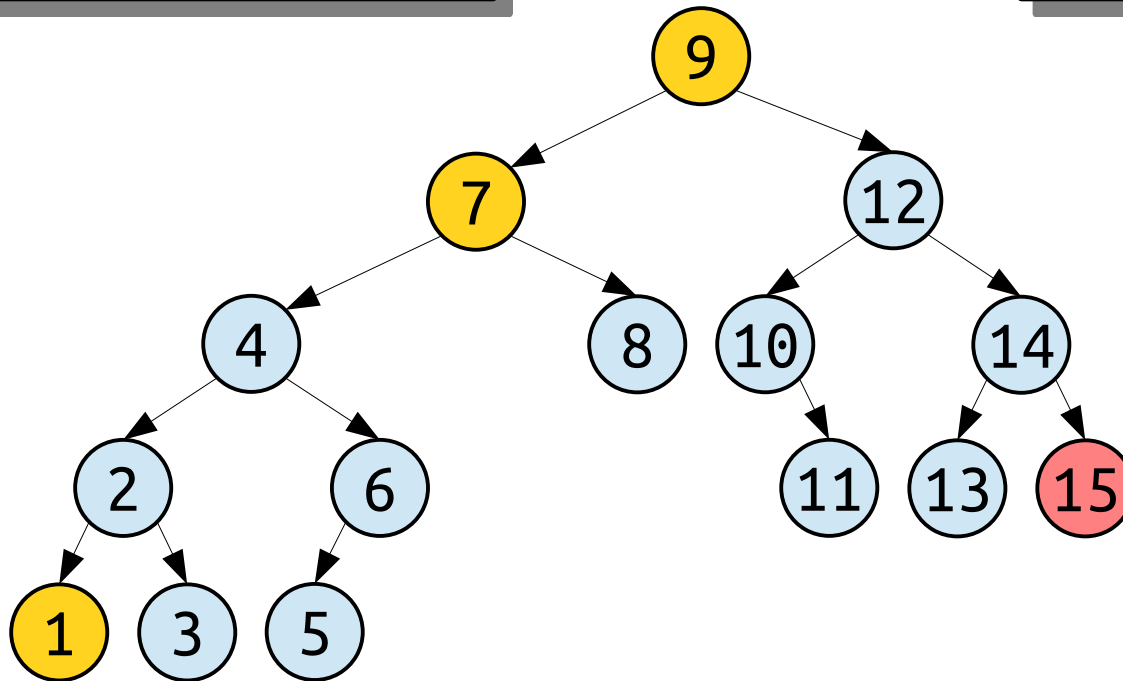
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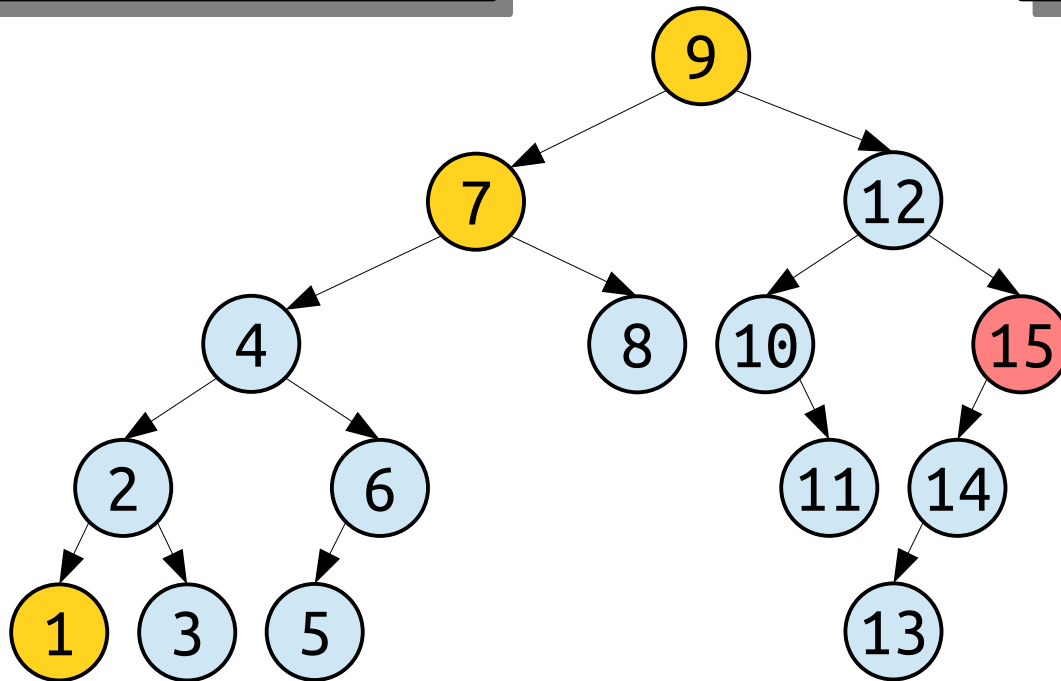
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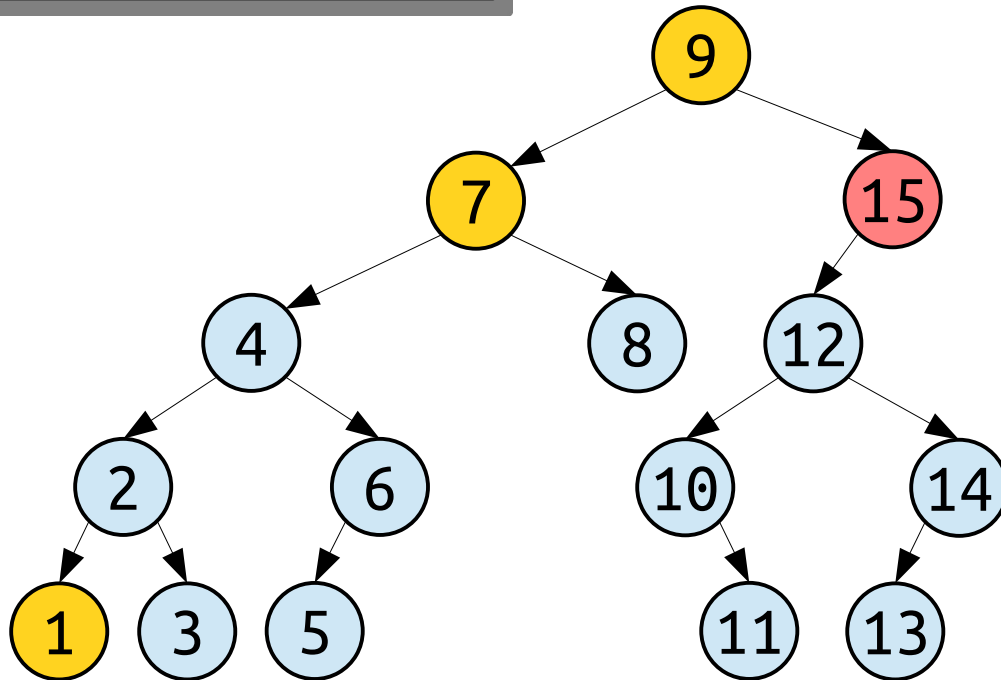
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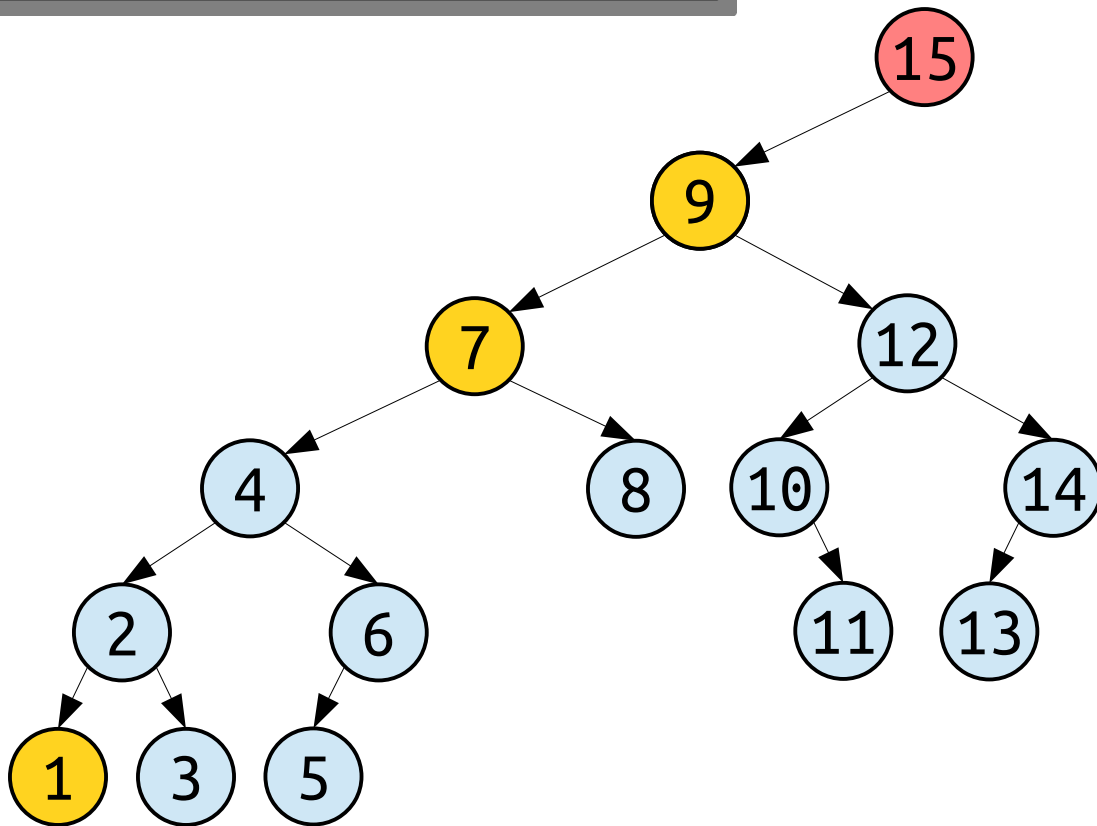
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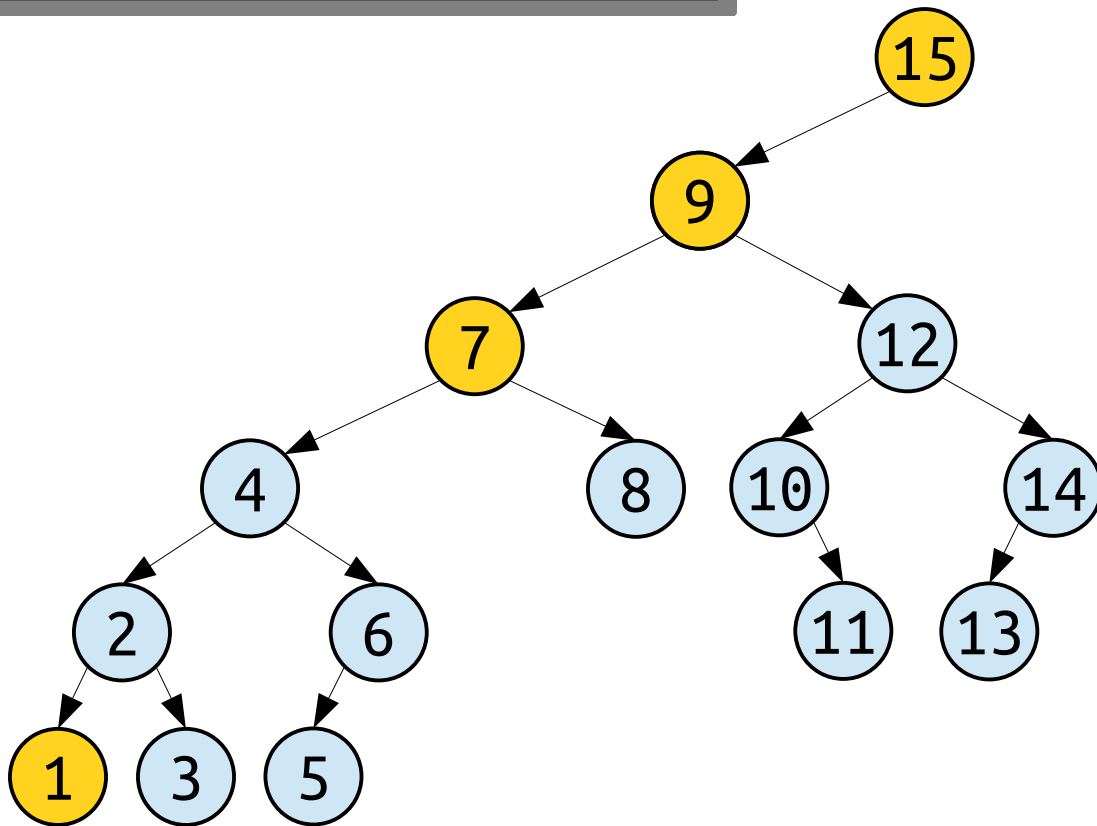


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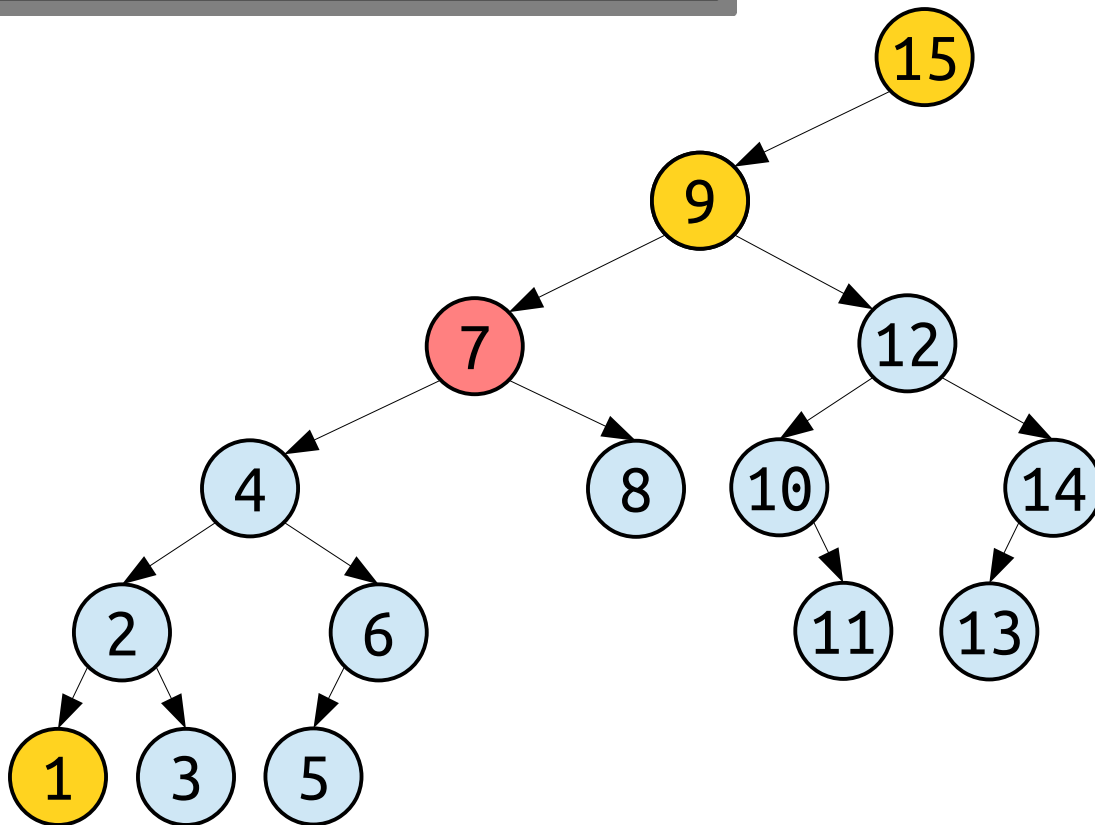
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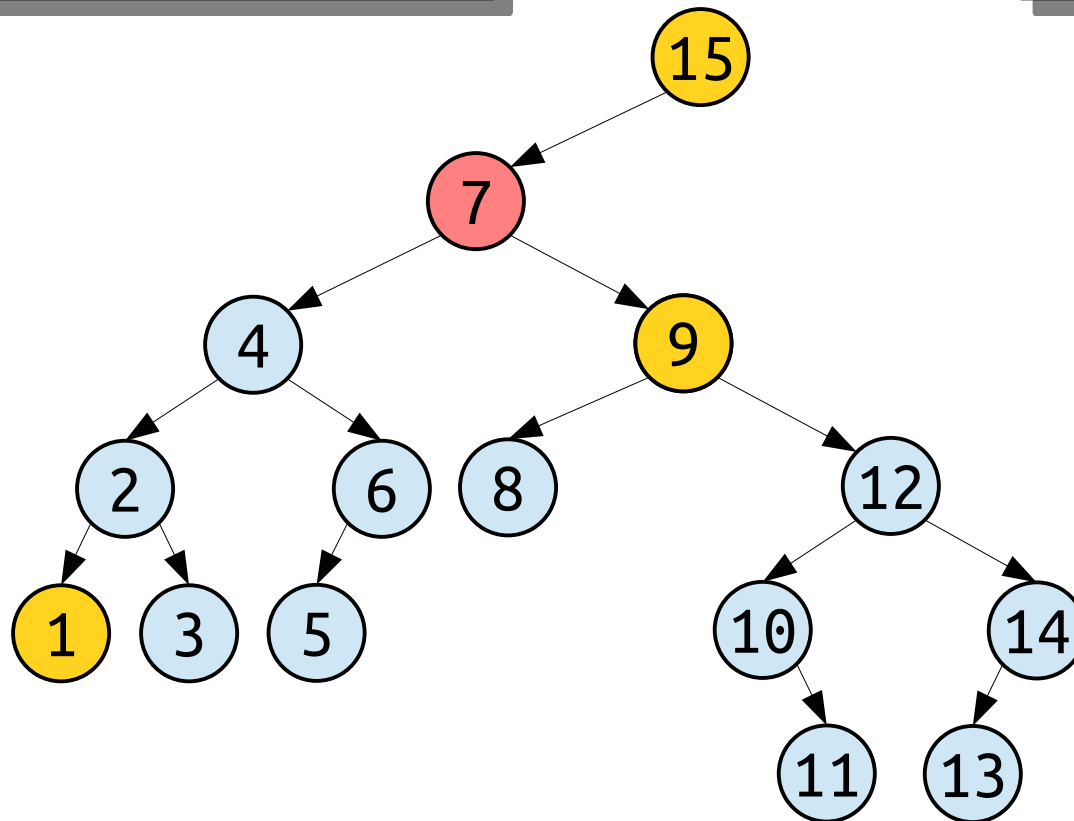
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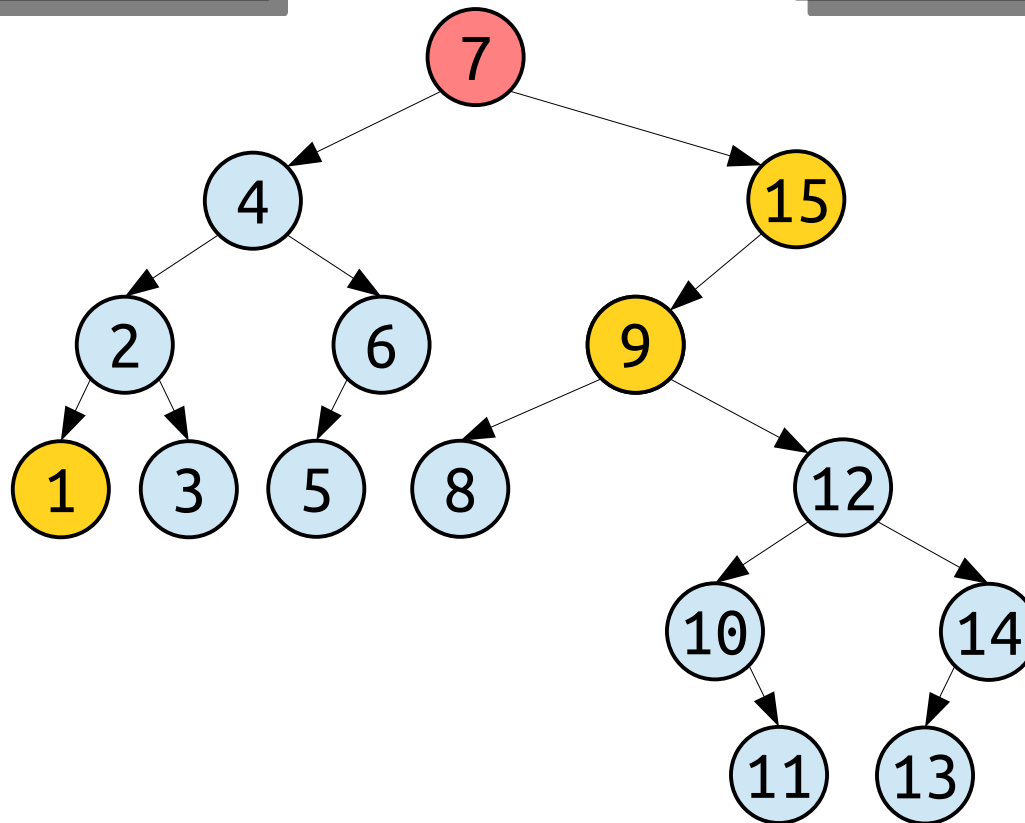
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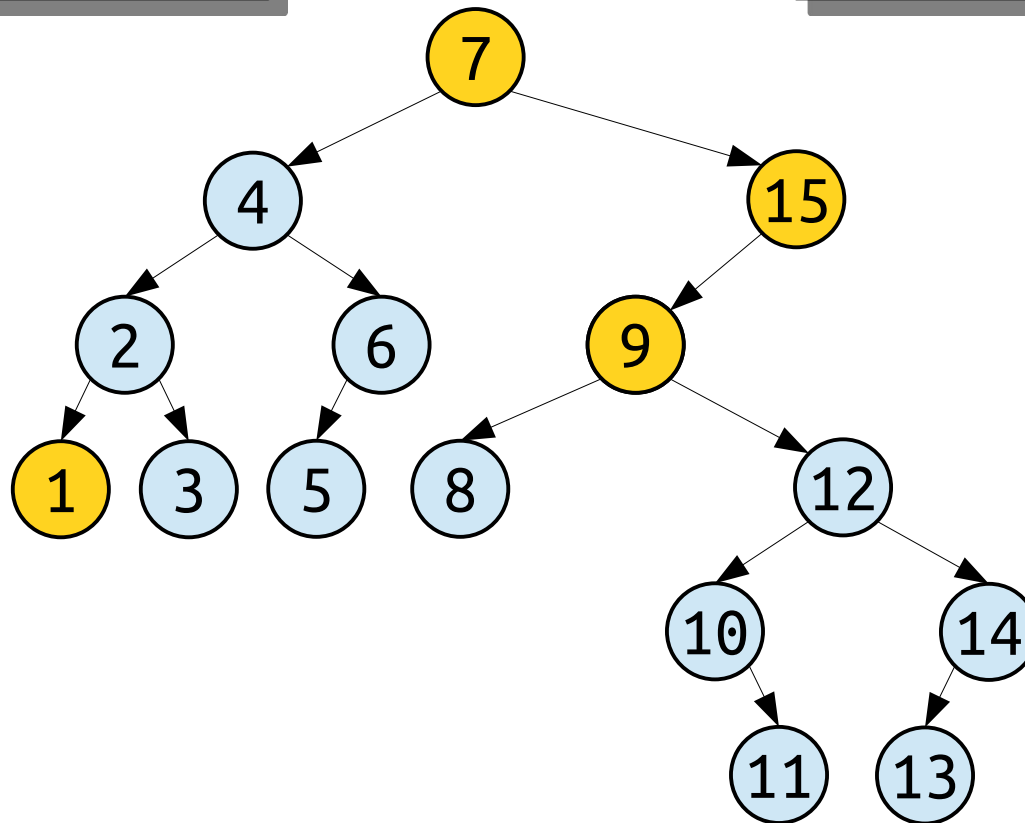
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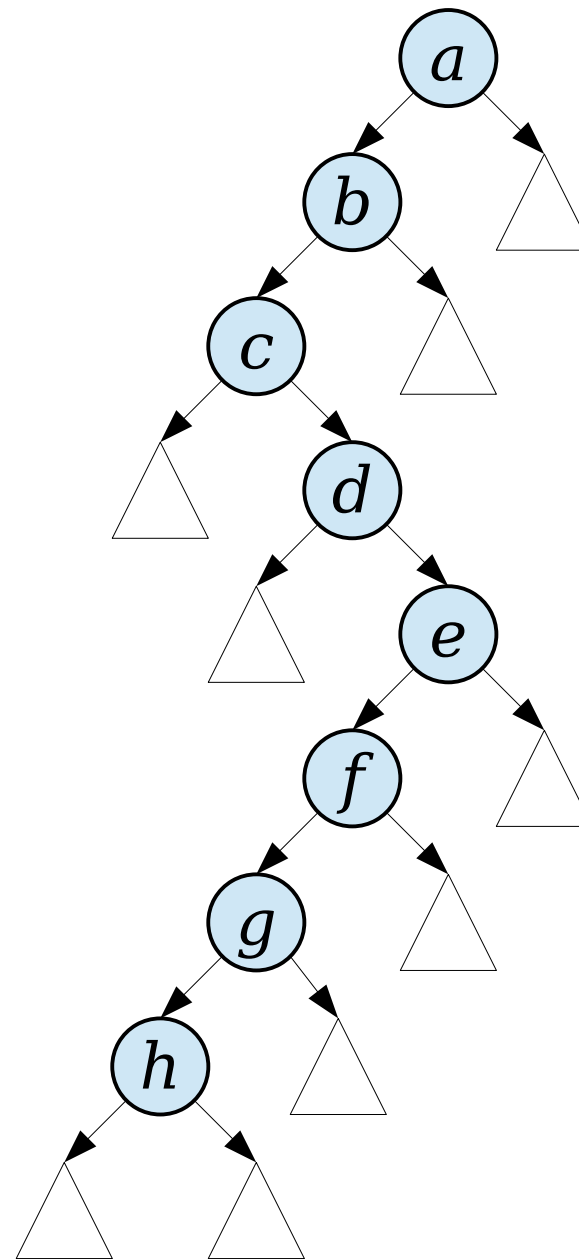
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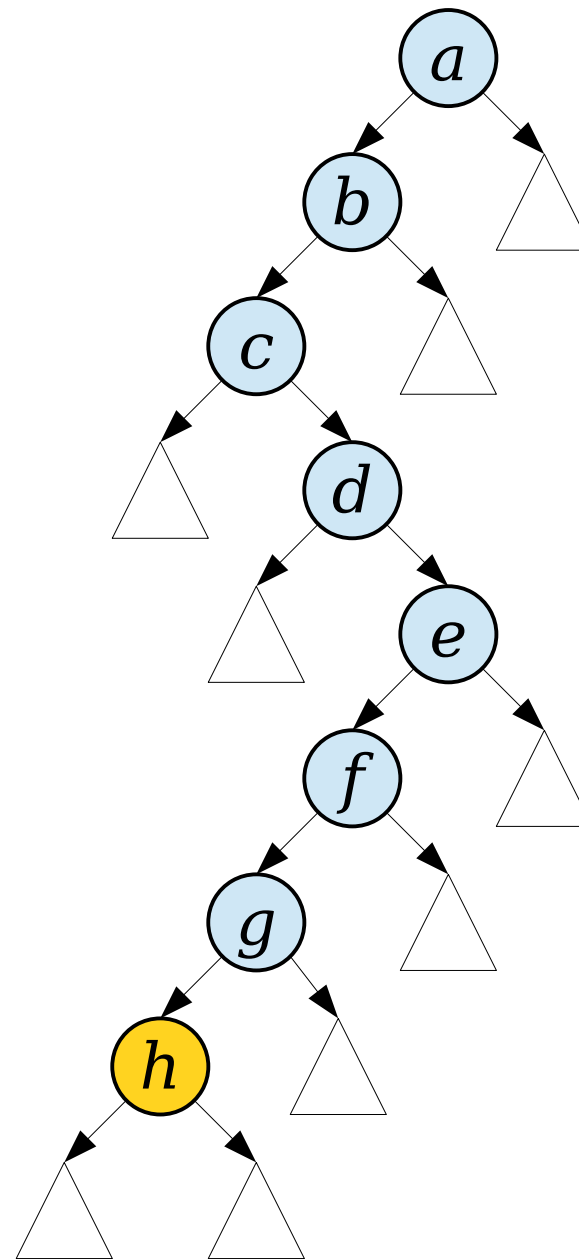
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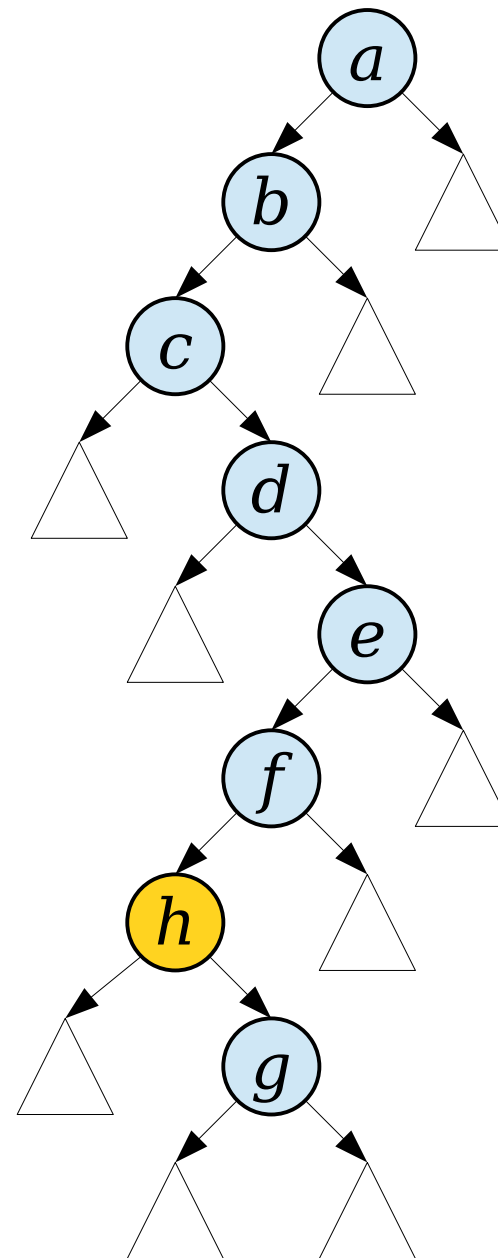
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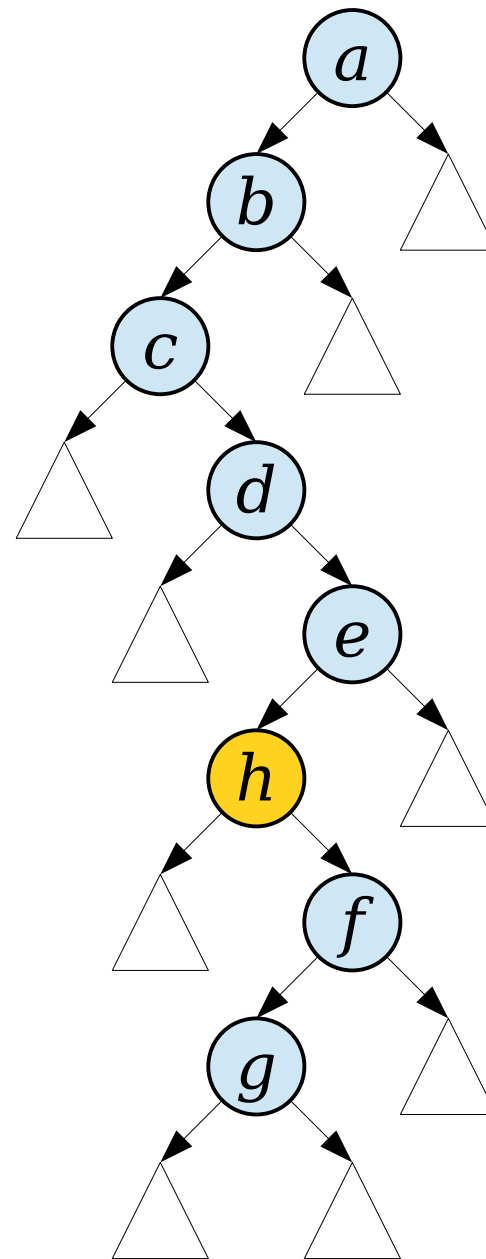
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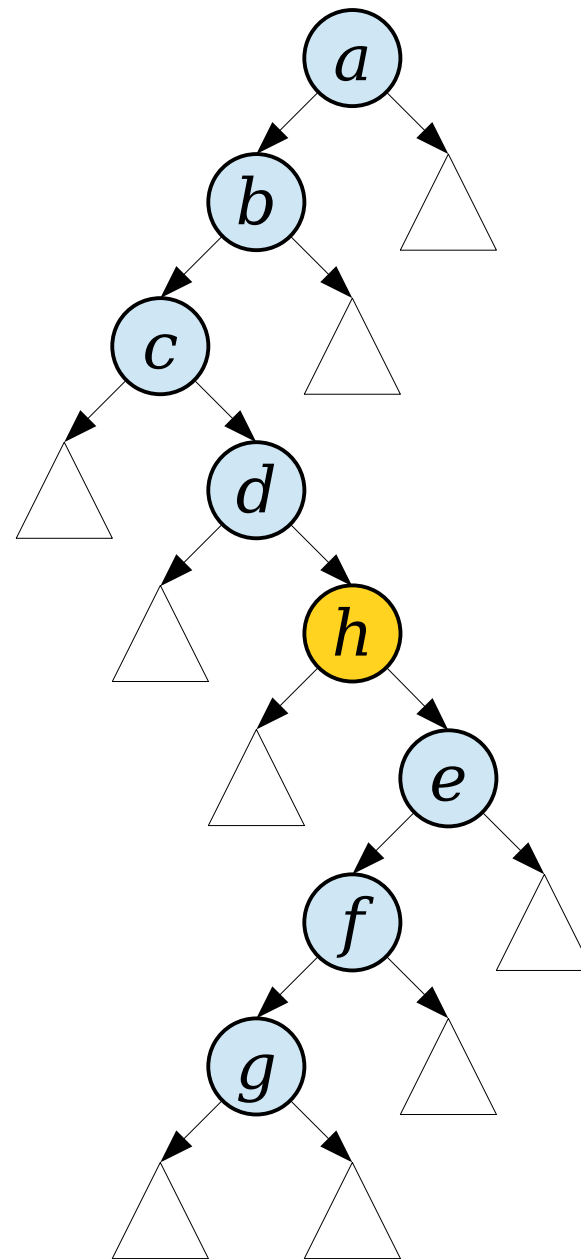


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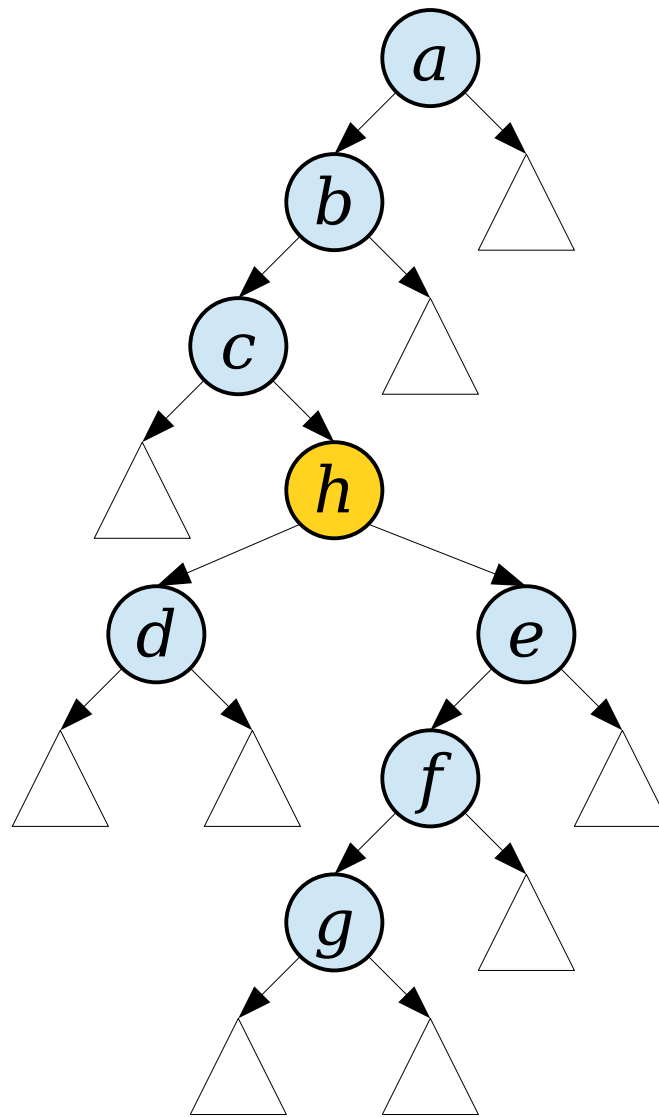
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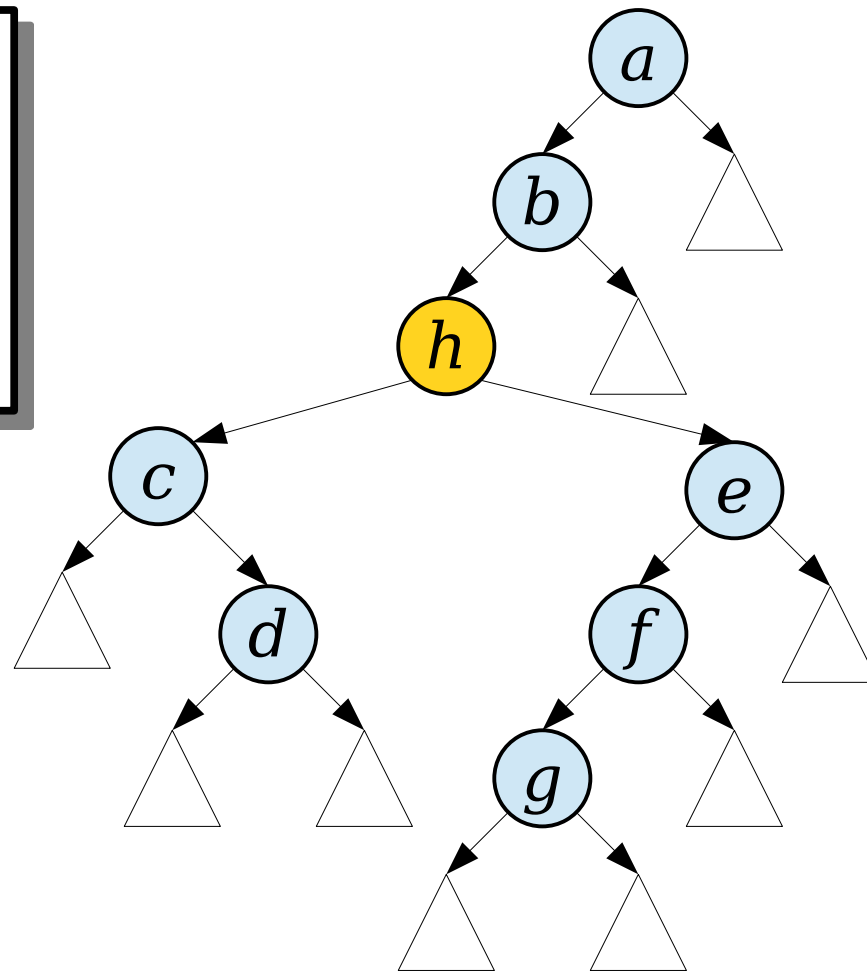
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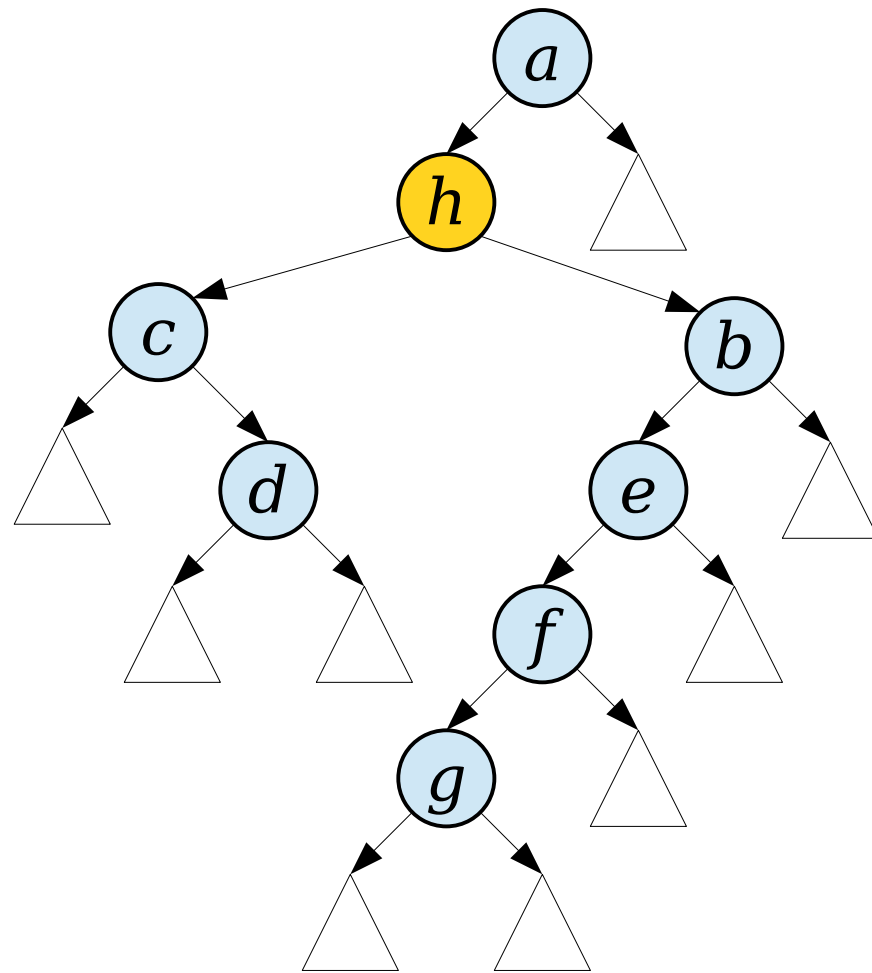
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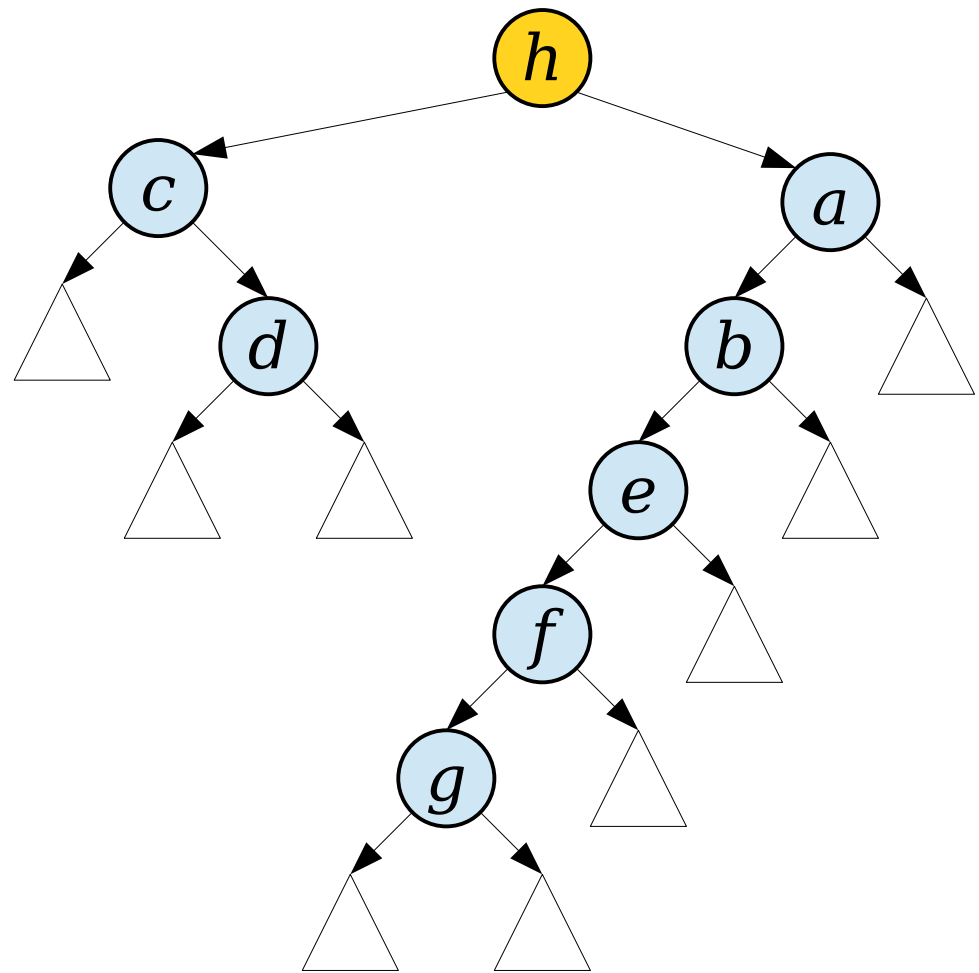
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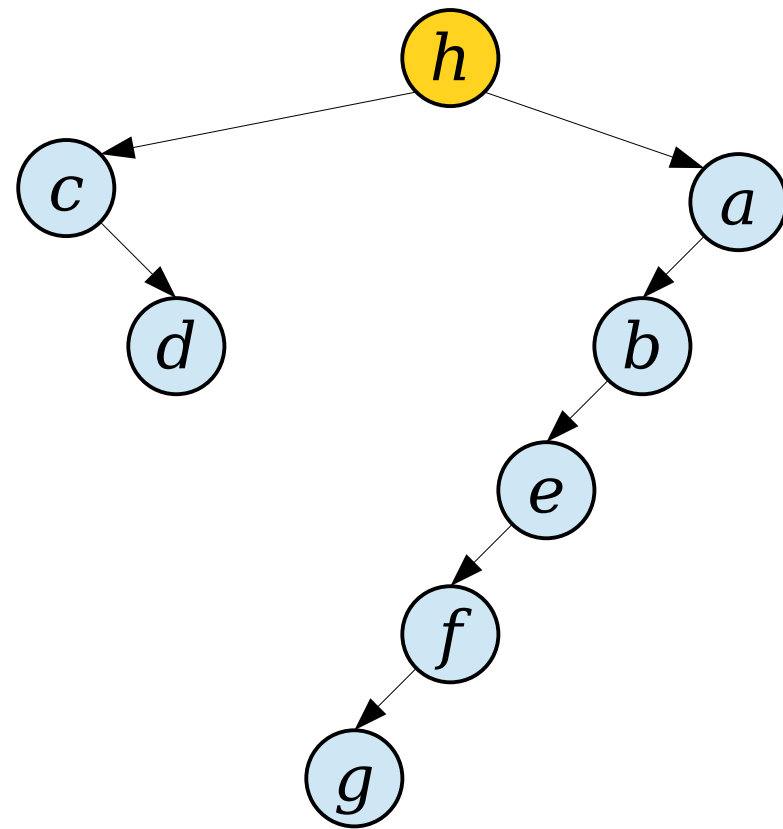
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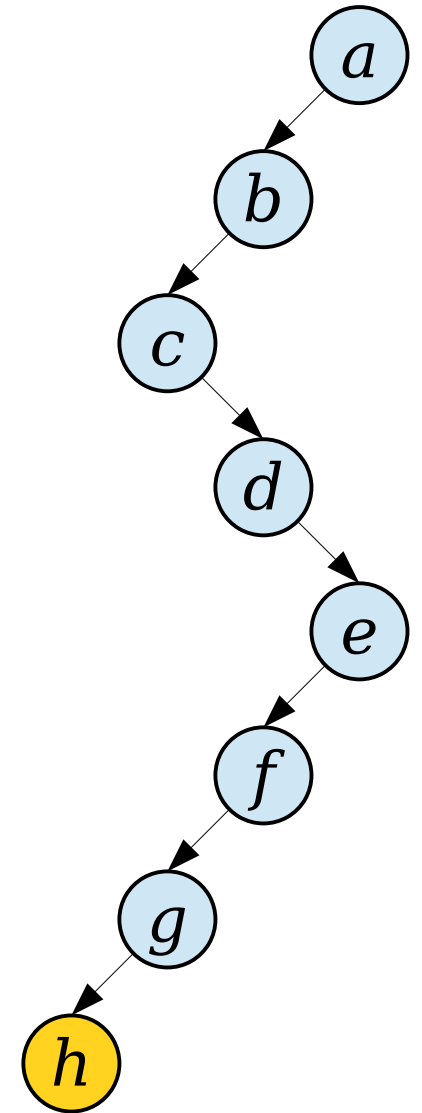
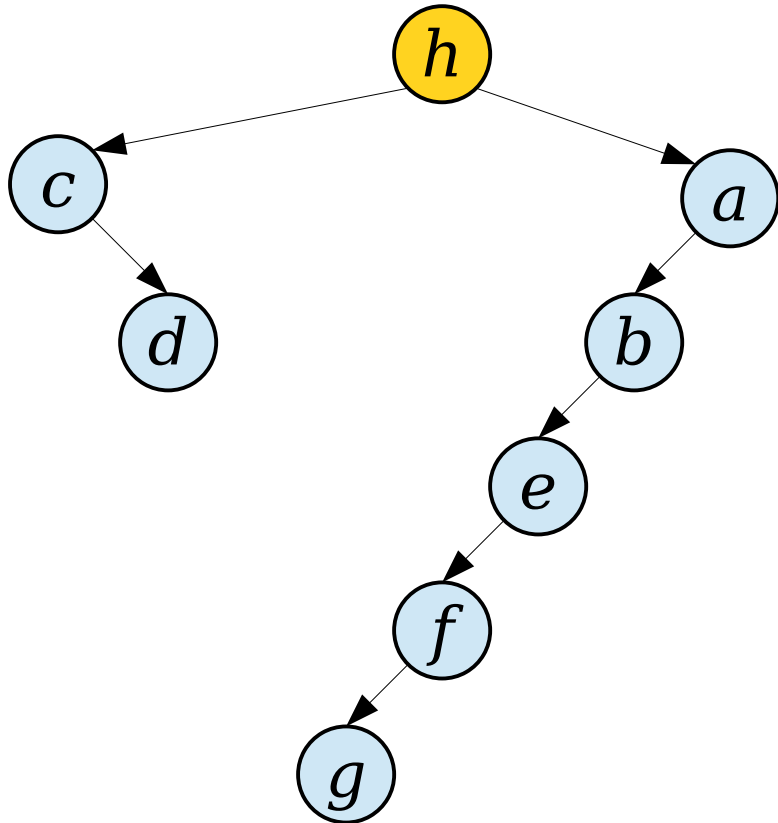
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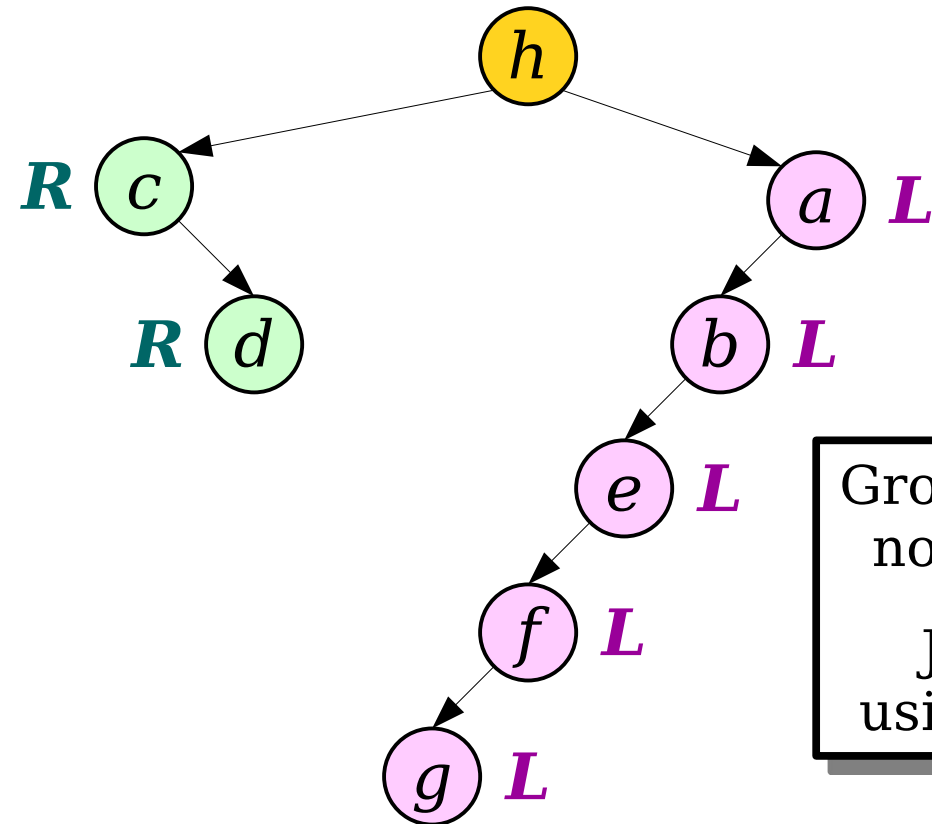
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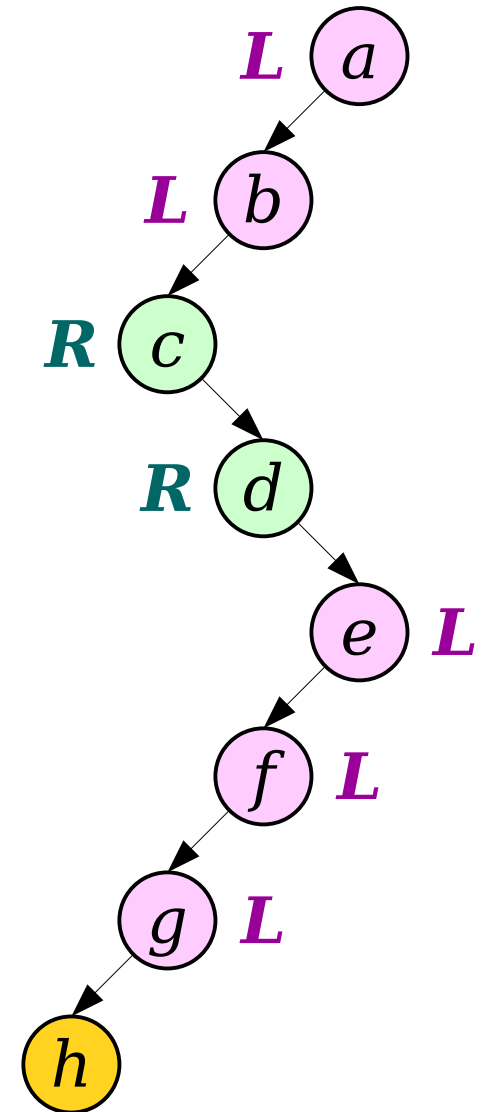


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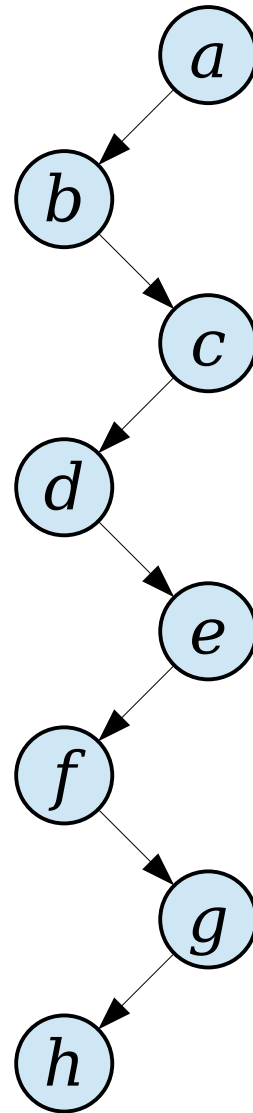
Group **R**-type and **L**-type nodes into two chains.

Join them together using the rotated node.



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**Observation 1:** This works really well on zig-zag-shaped trees.

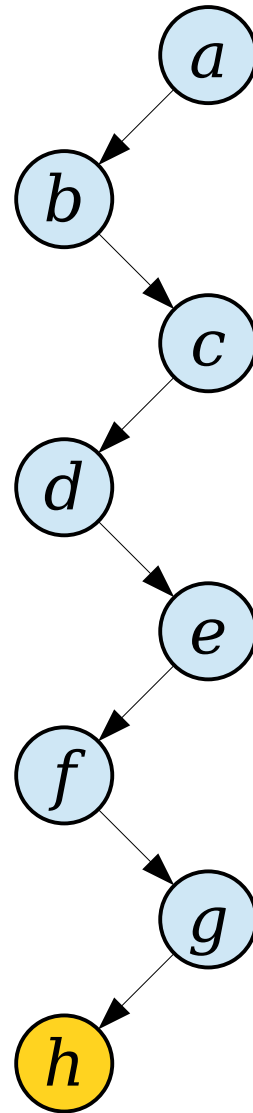


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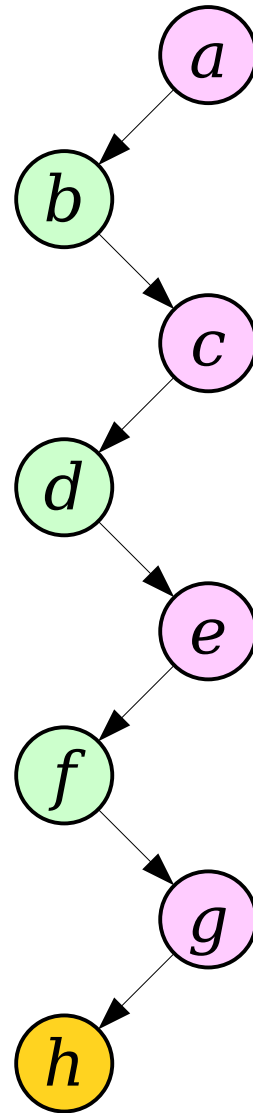


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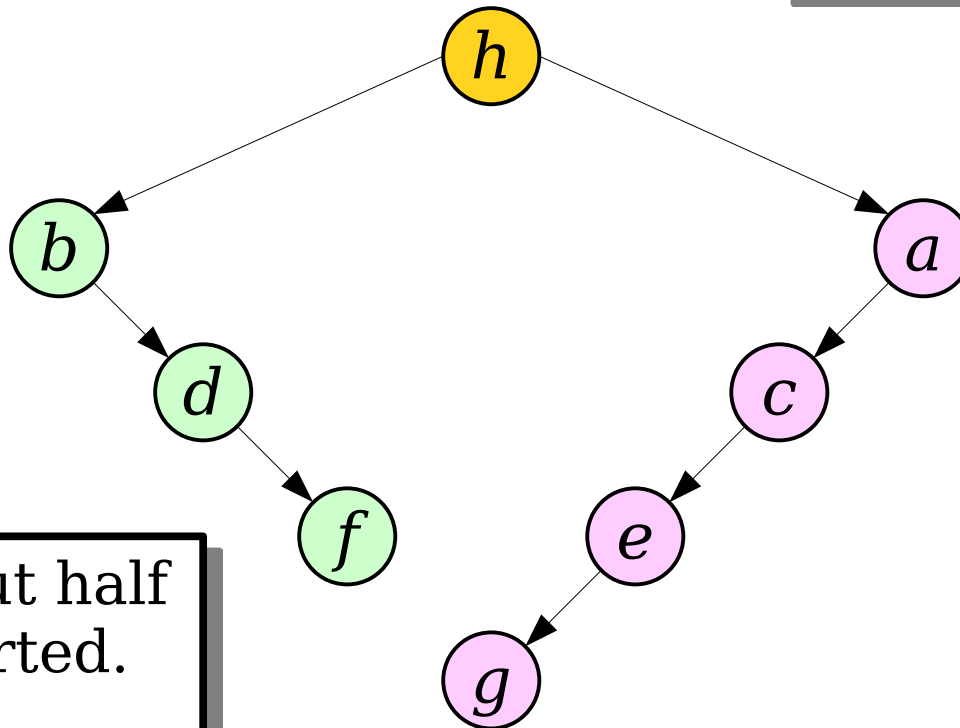
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This tree is about half as tall as it started.

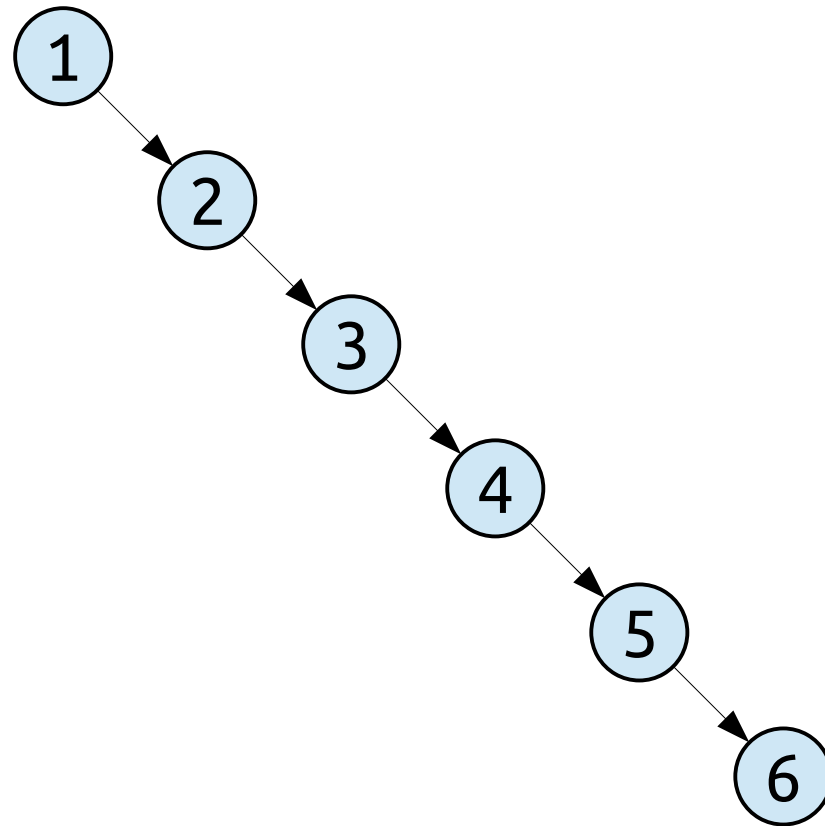
Most nodes on the access path are much closer to the root.

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**Observation 2:** This does not work well at all on long chains.

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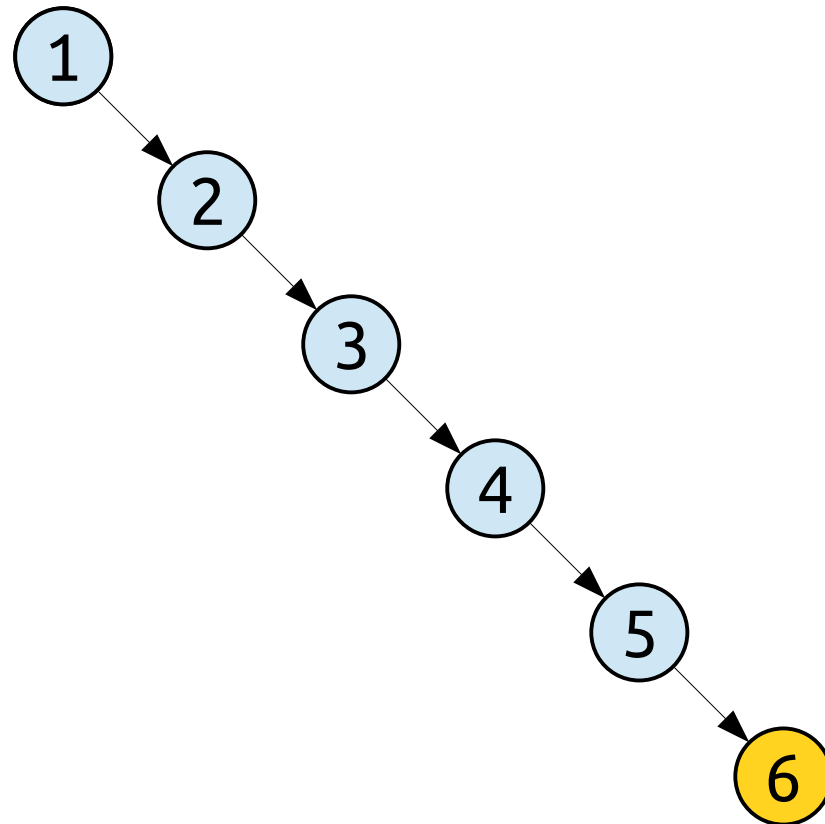
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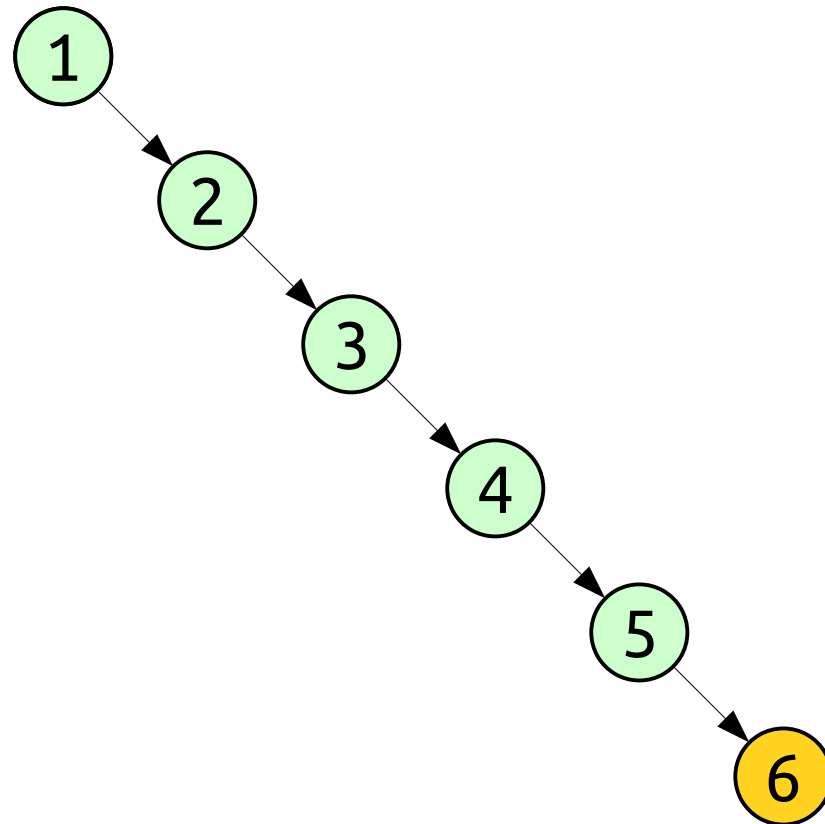
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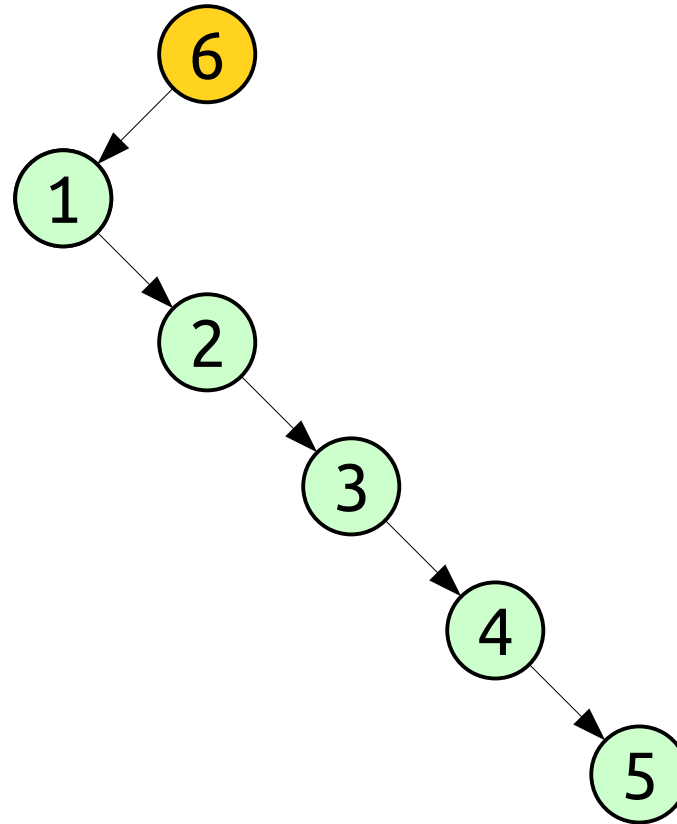
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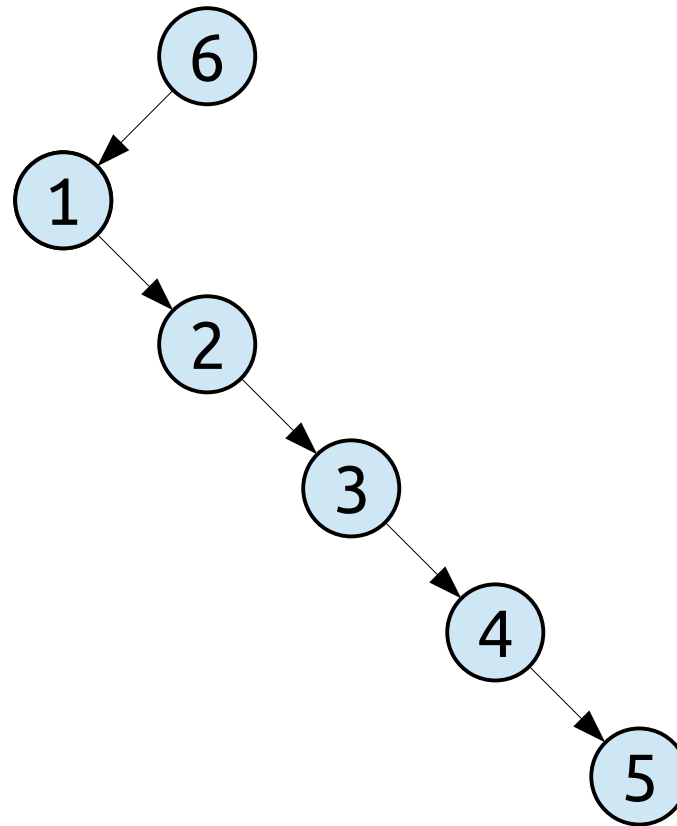
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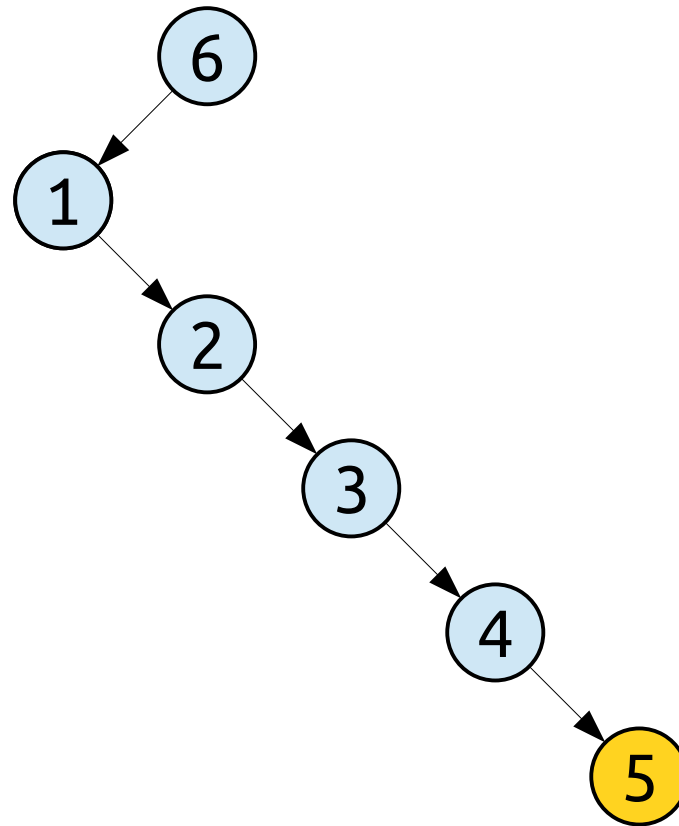
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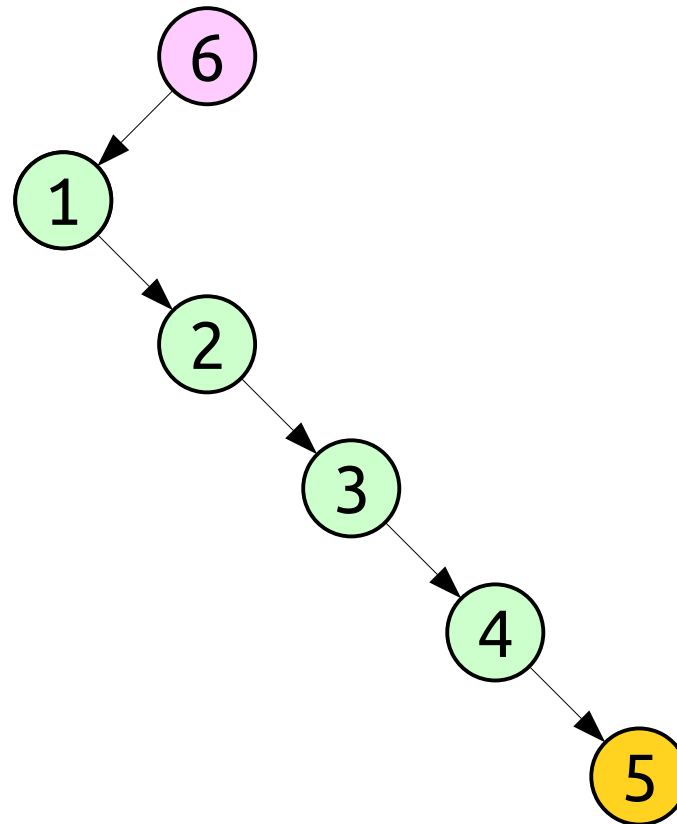
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**Question:** Does rotating each accessed key to the root guarantee good overall performance?

**Observation 2:** This does not work well at all on long chains.

Group **R**-type and **L**-type nodes into two chains.

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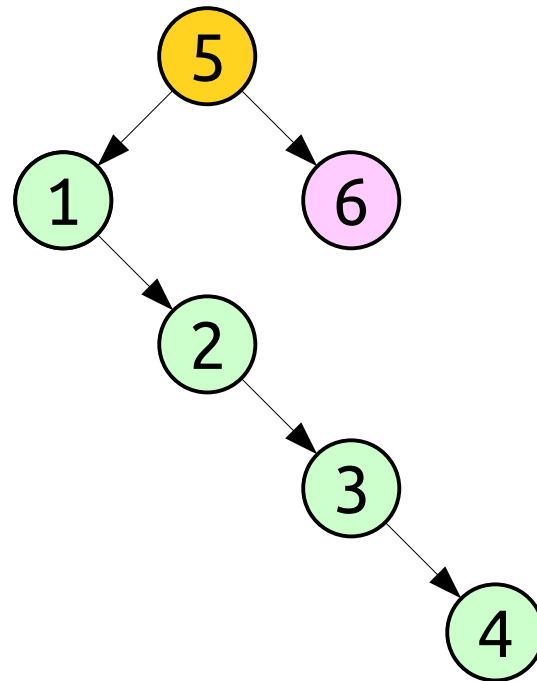
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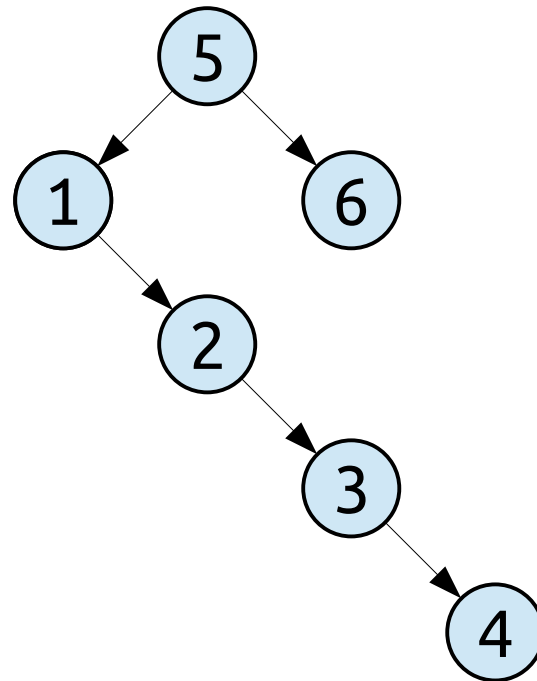
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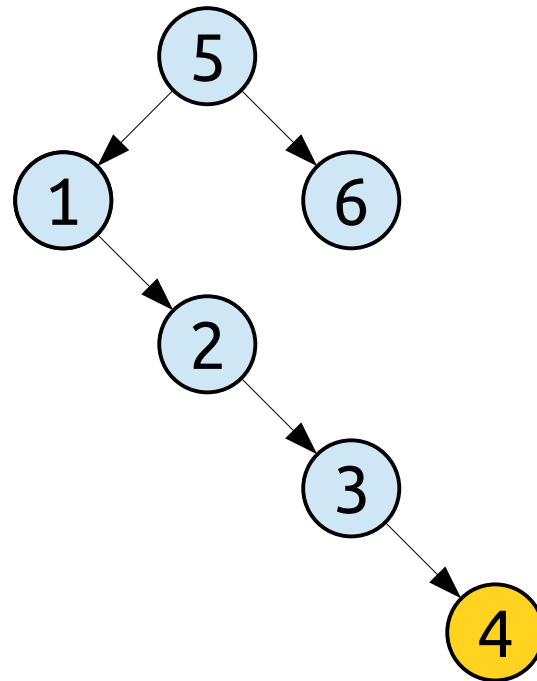
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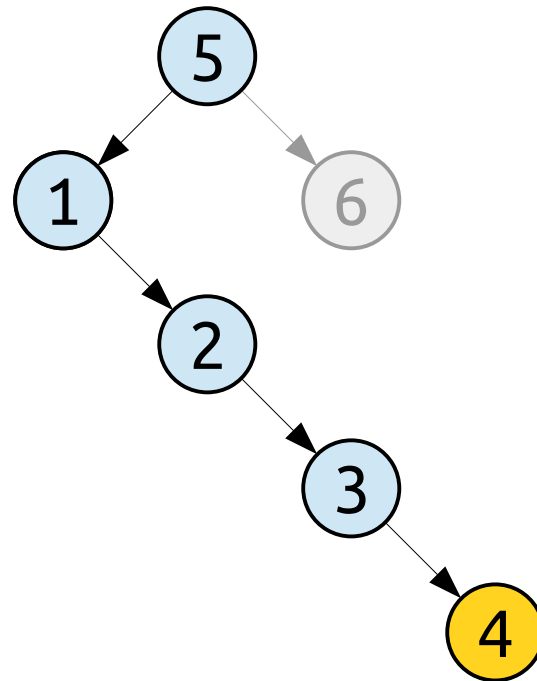
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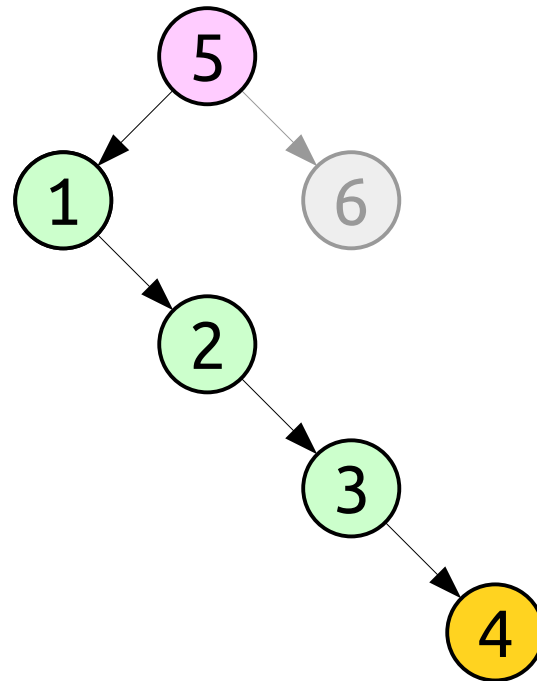
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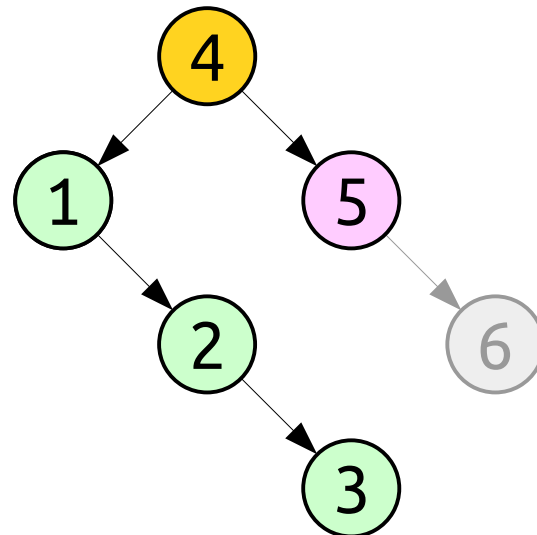
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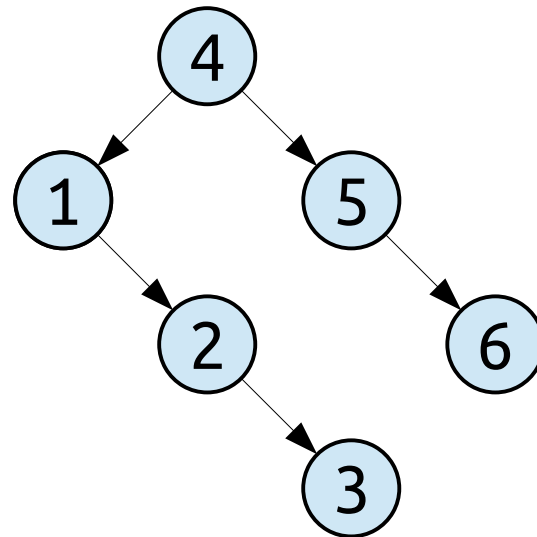


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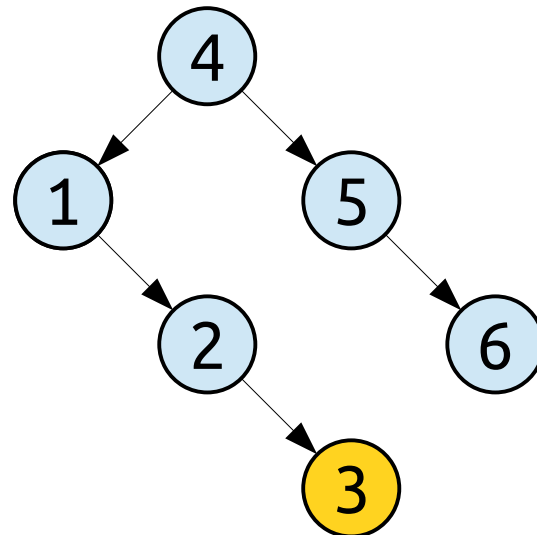
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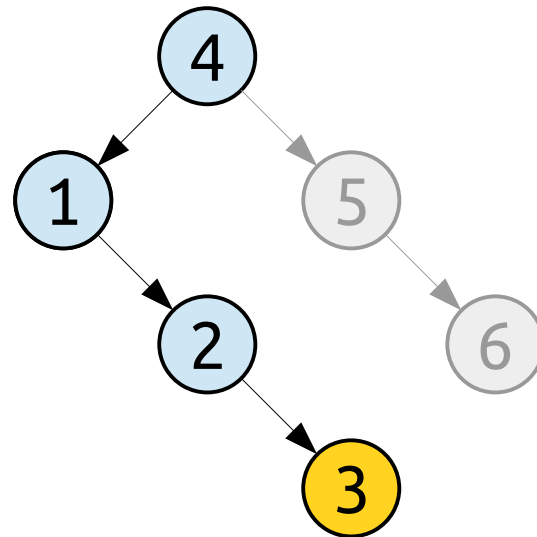
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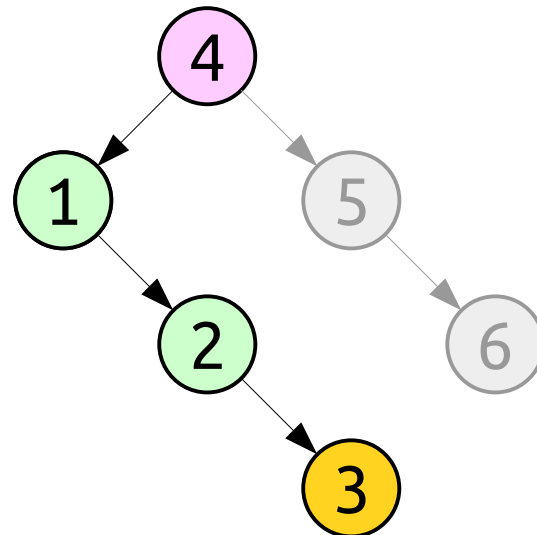
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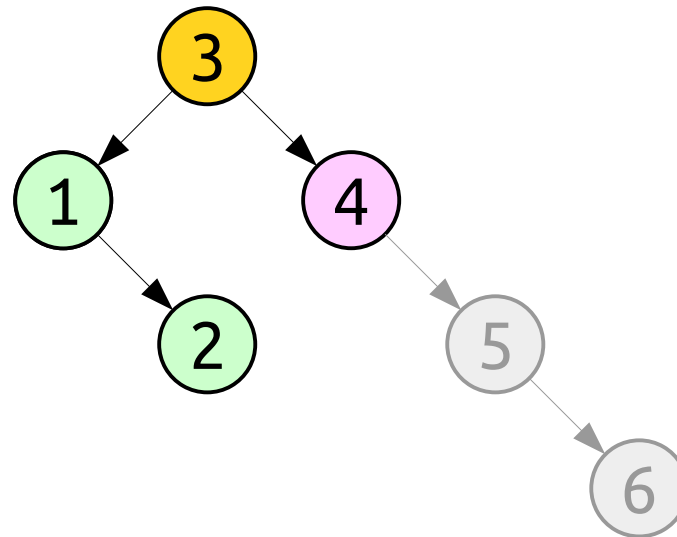
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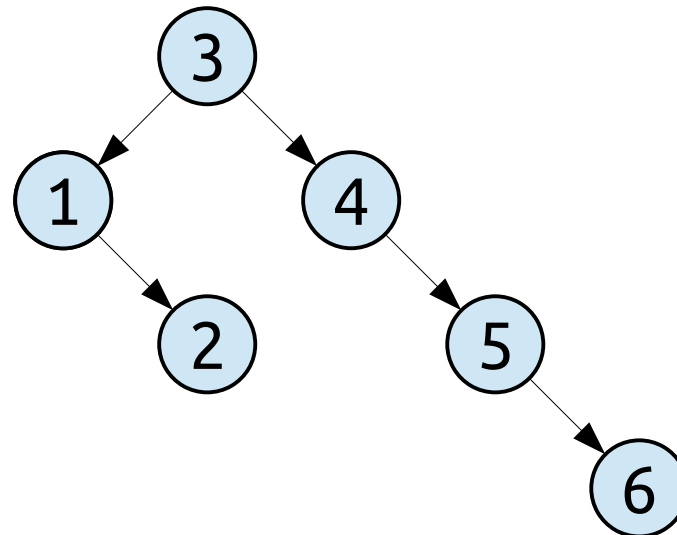


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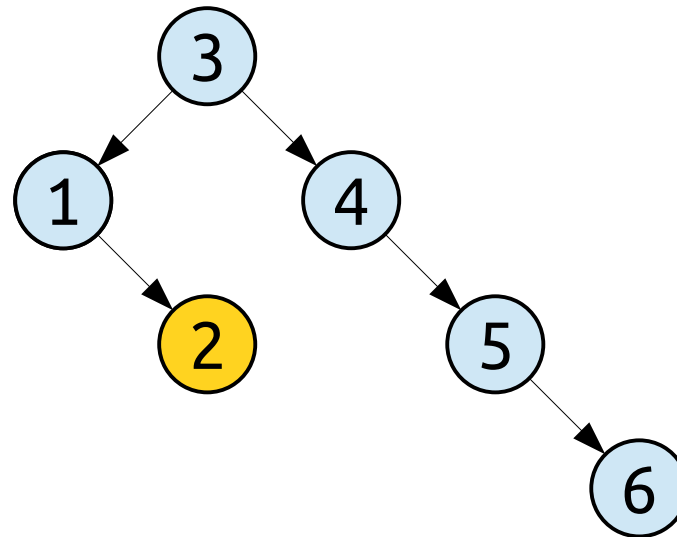
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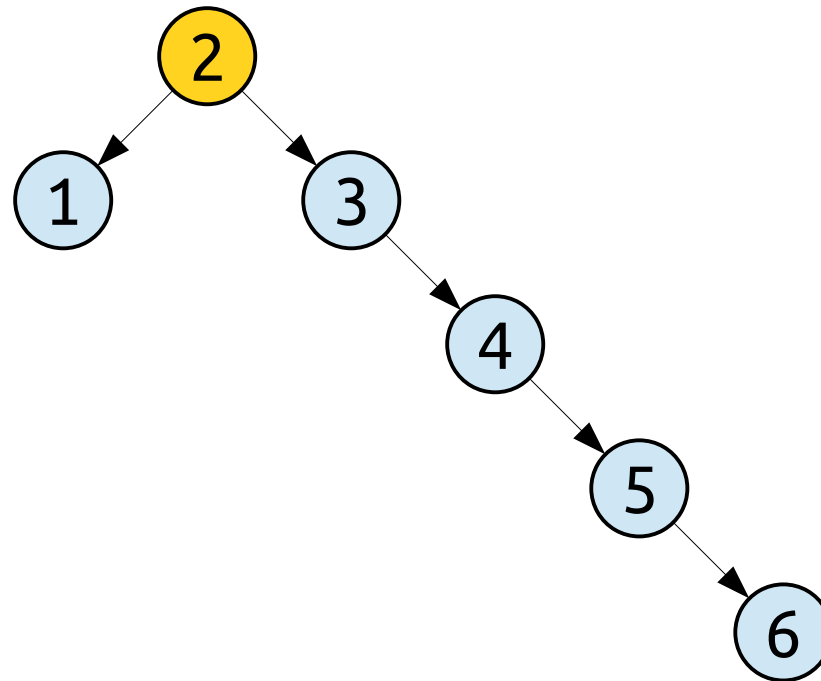
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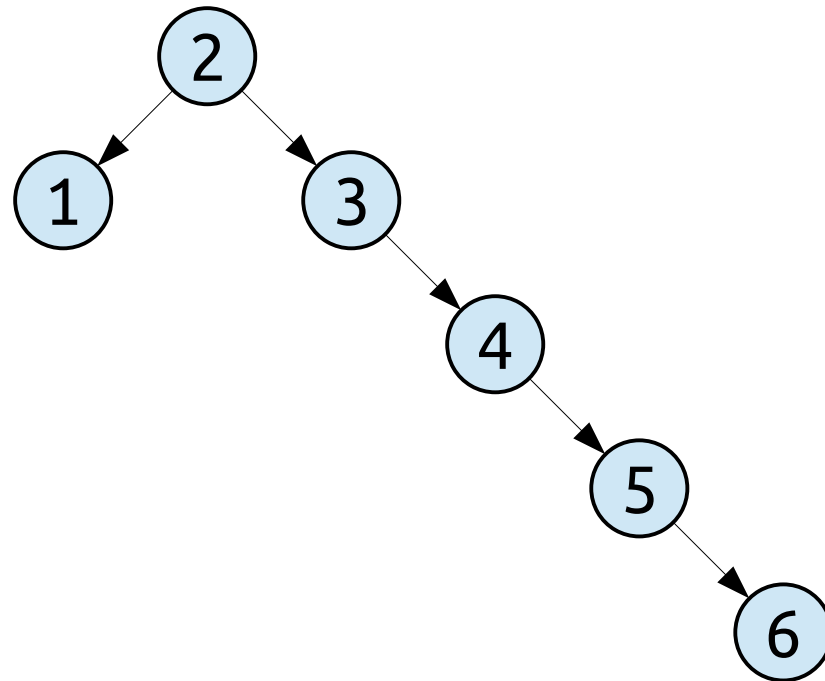
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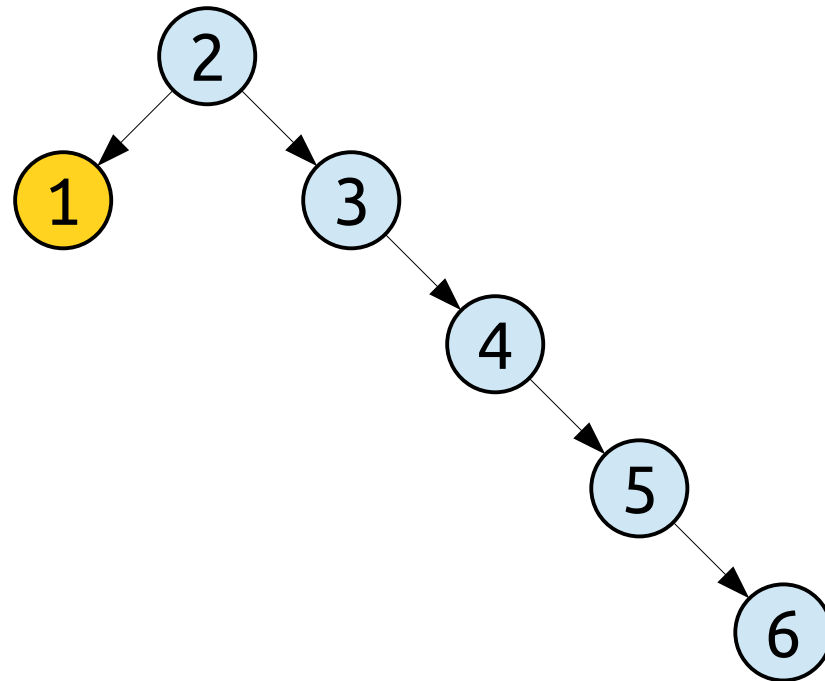
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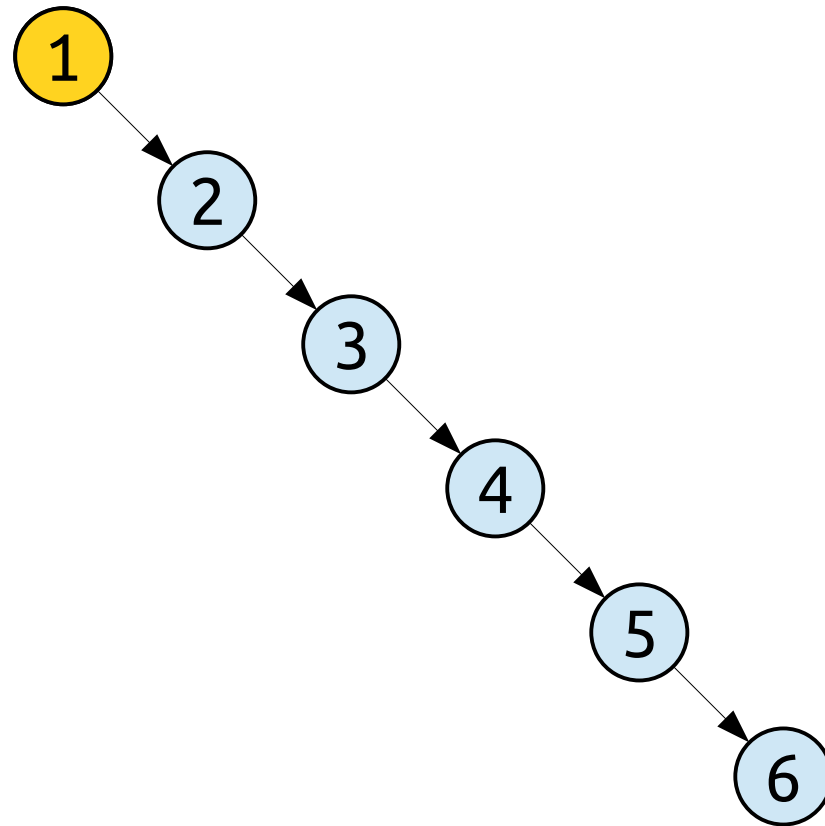
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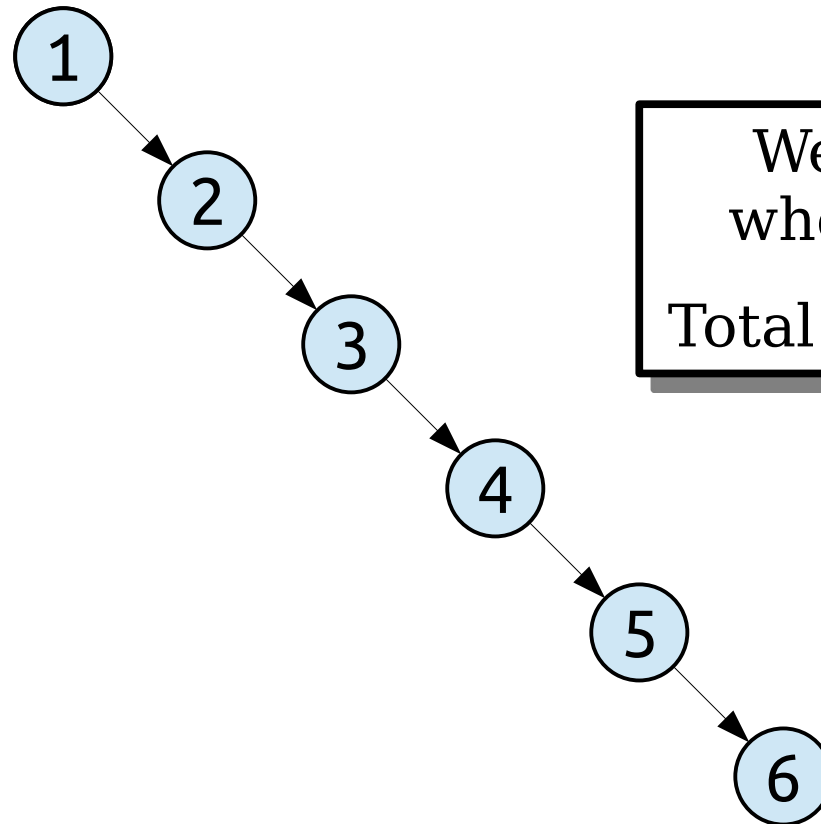
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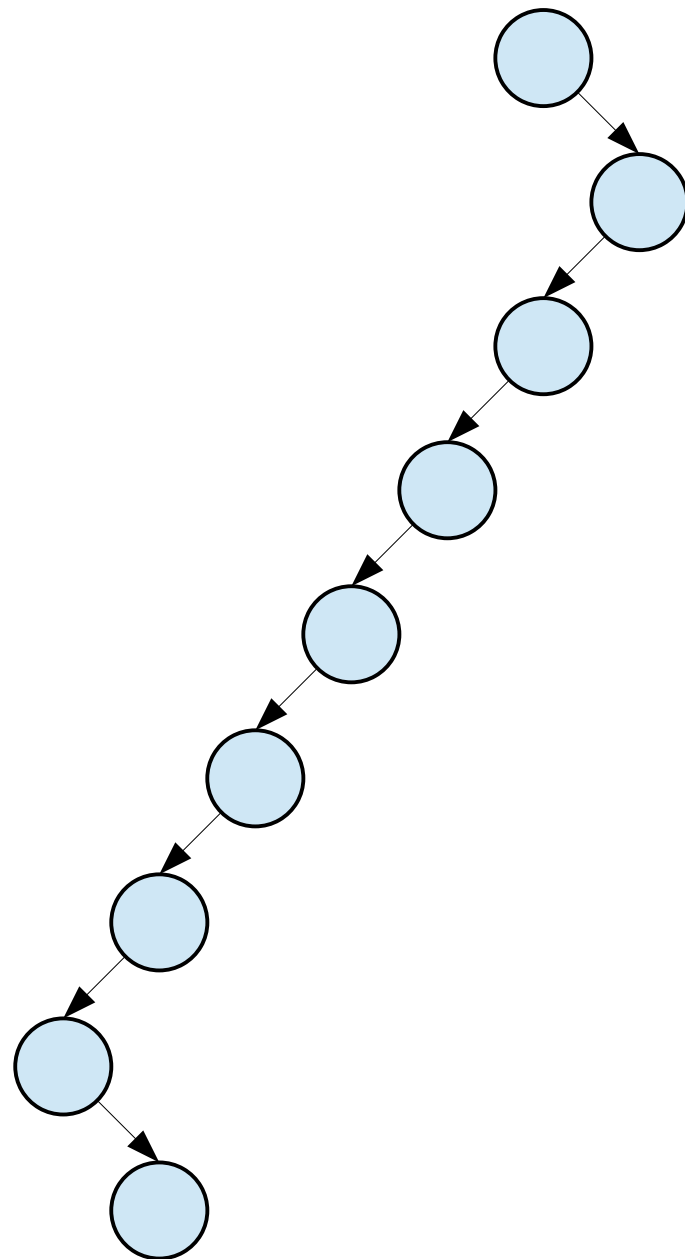
We're right back where we started!

Total rotations:  $\Theta(n^2)$ .

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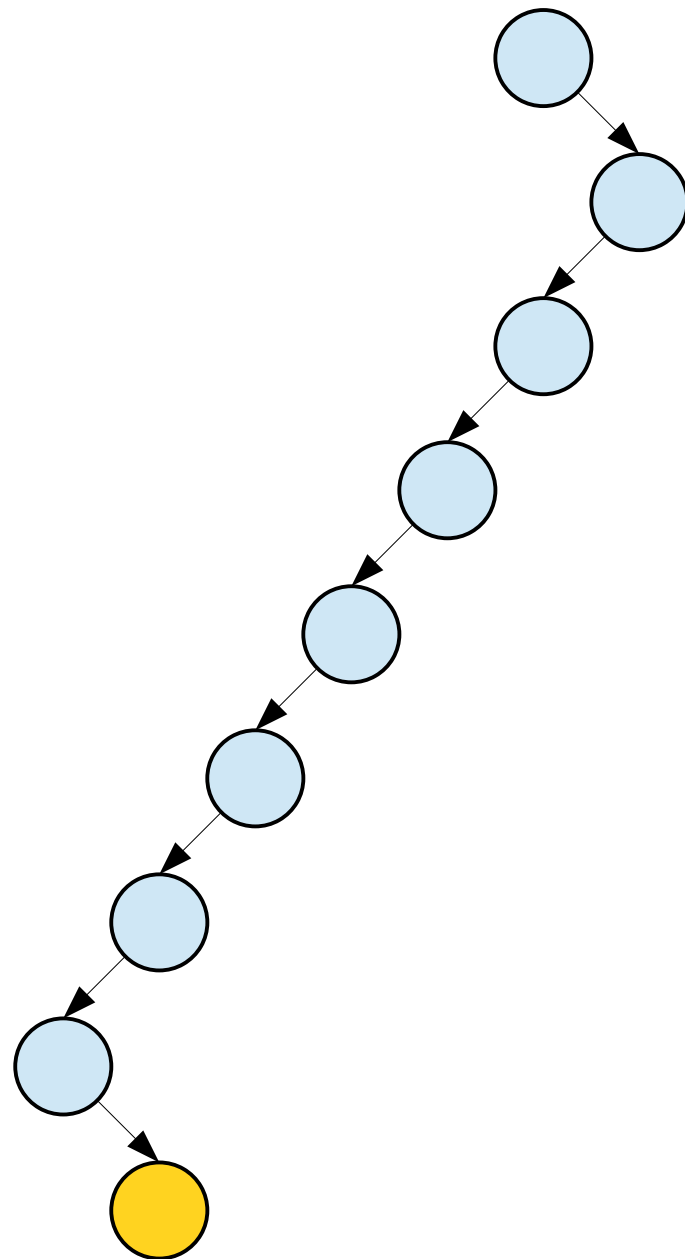
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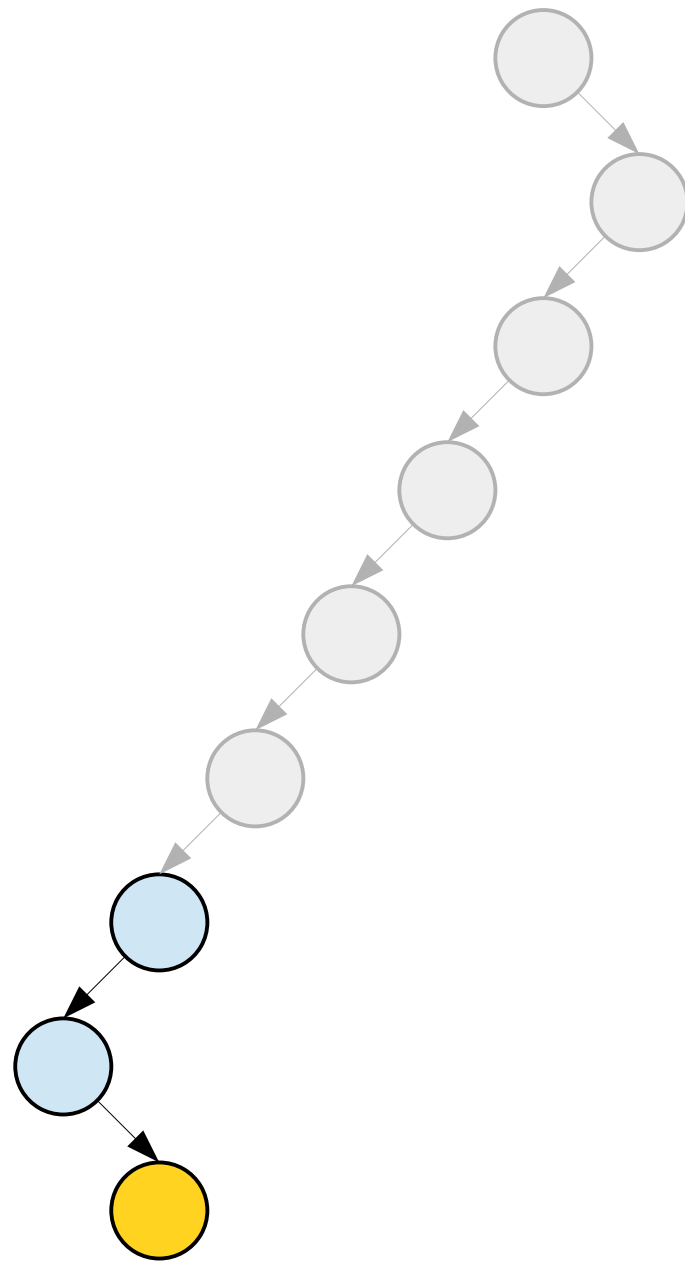


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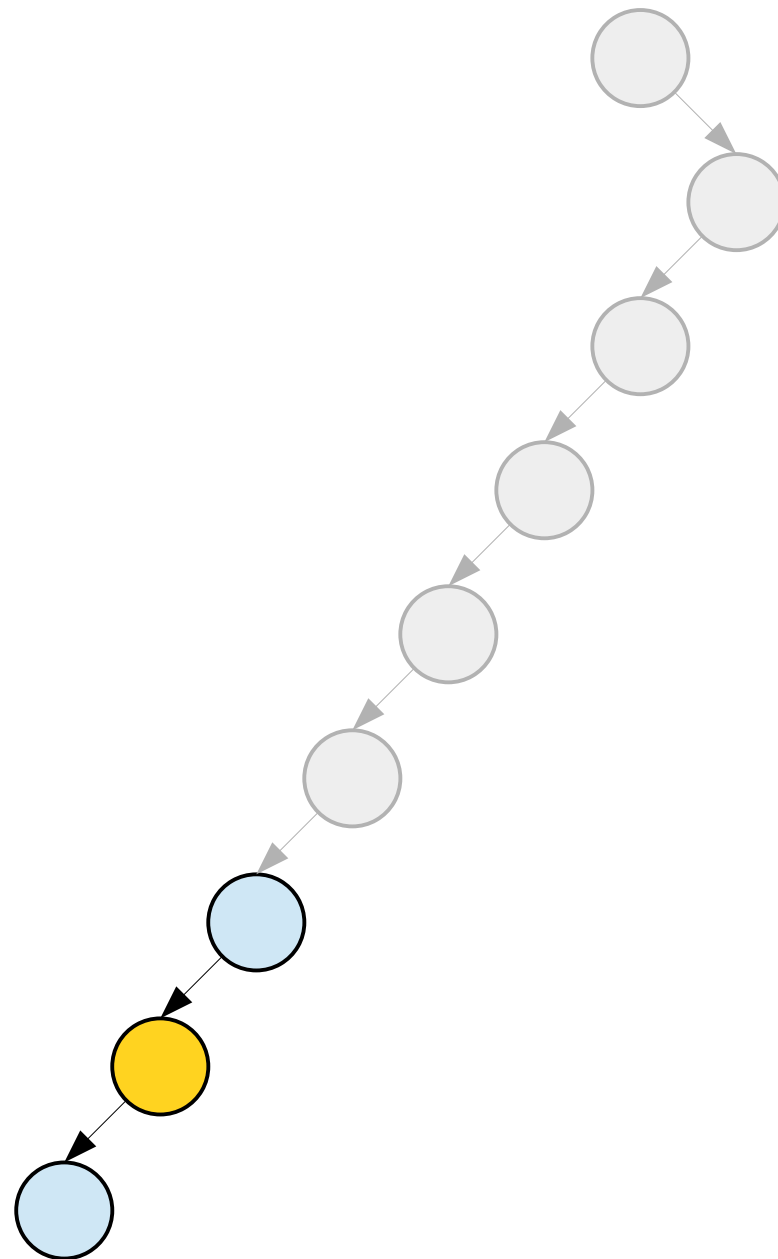
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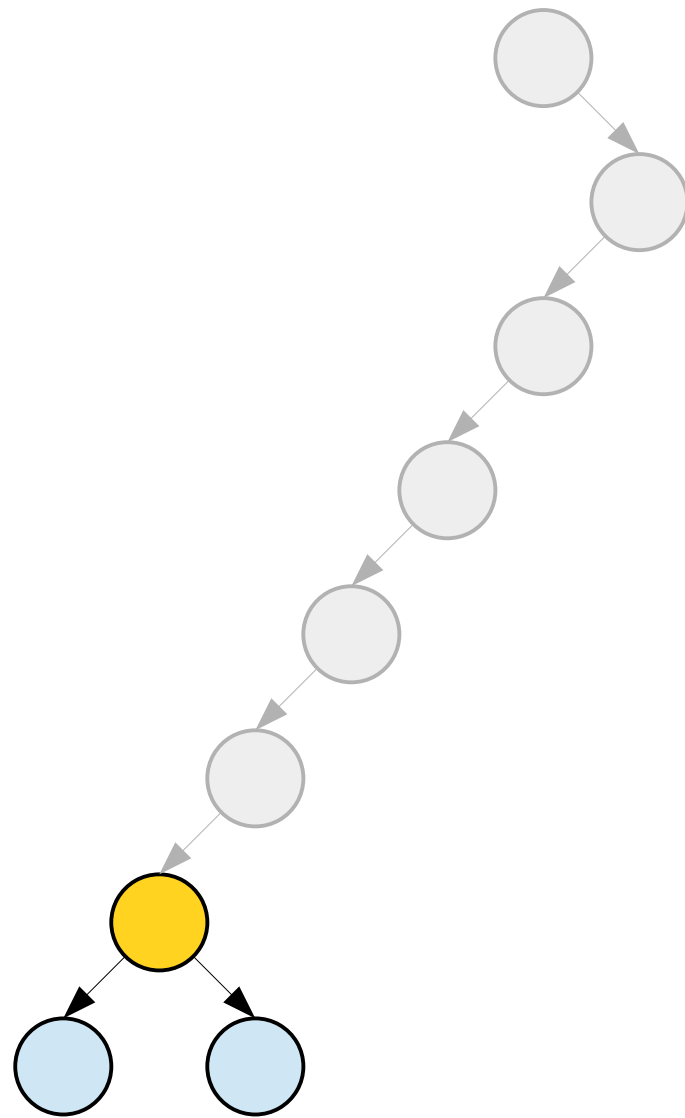
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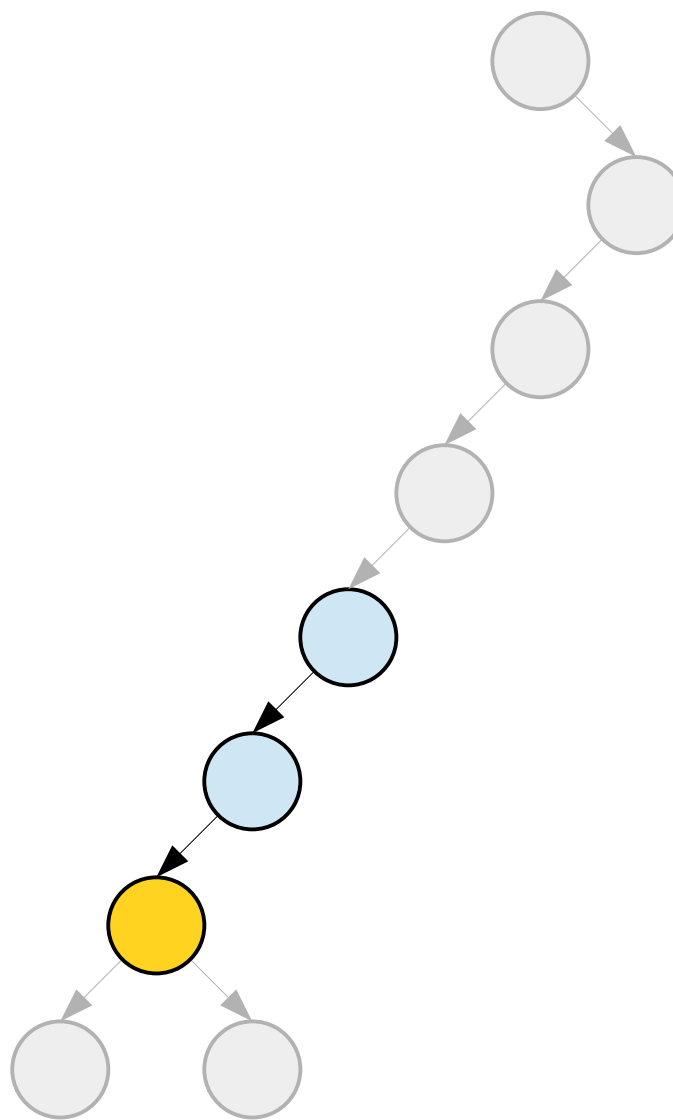
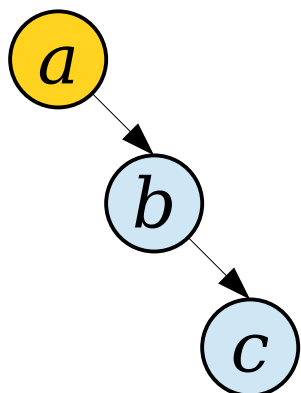
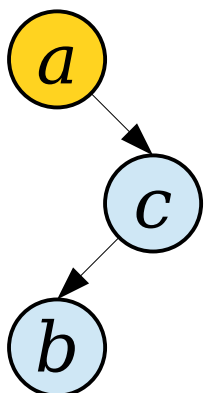
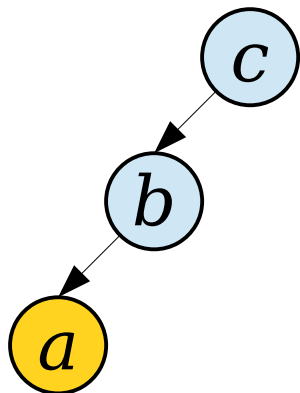
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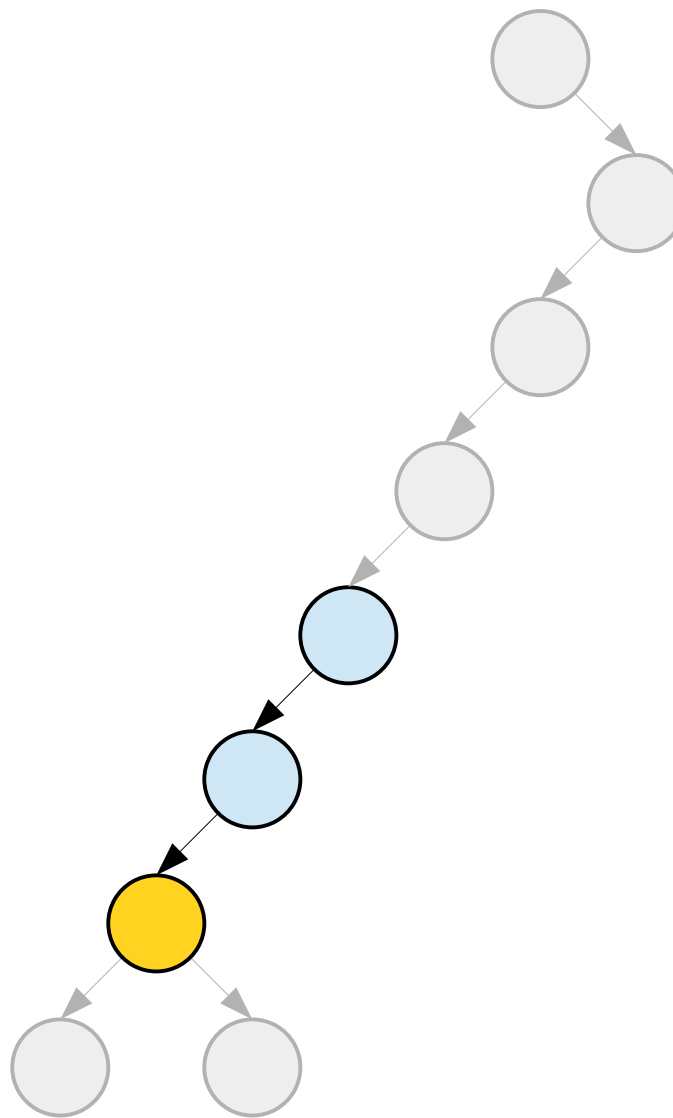
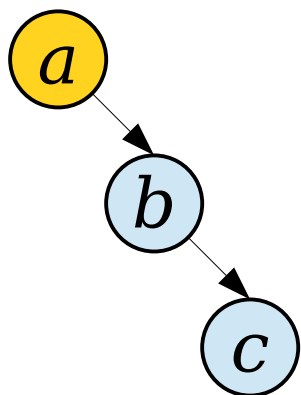
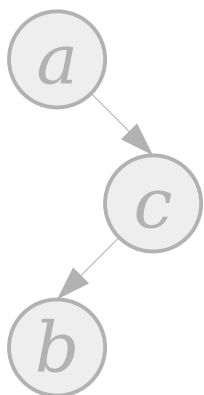
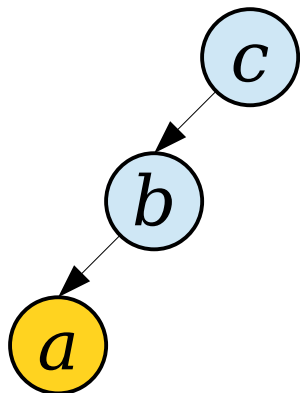
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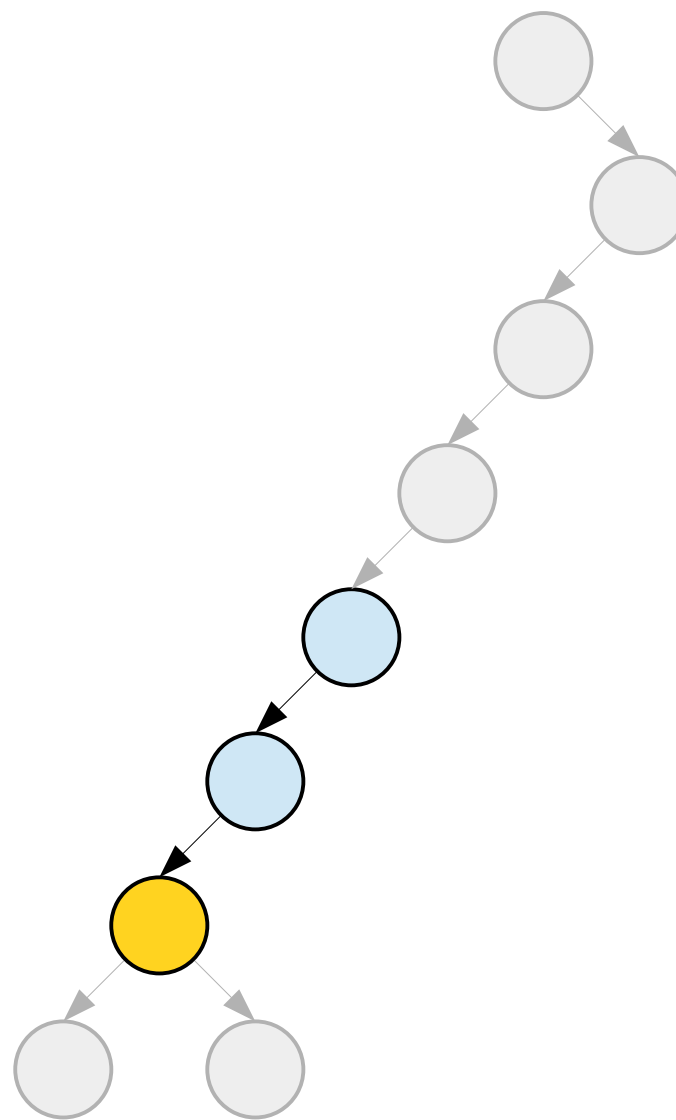
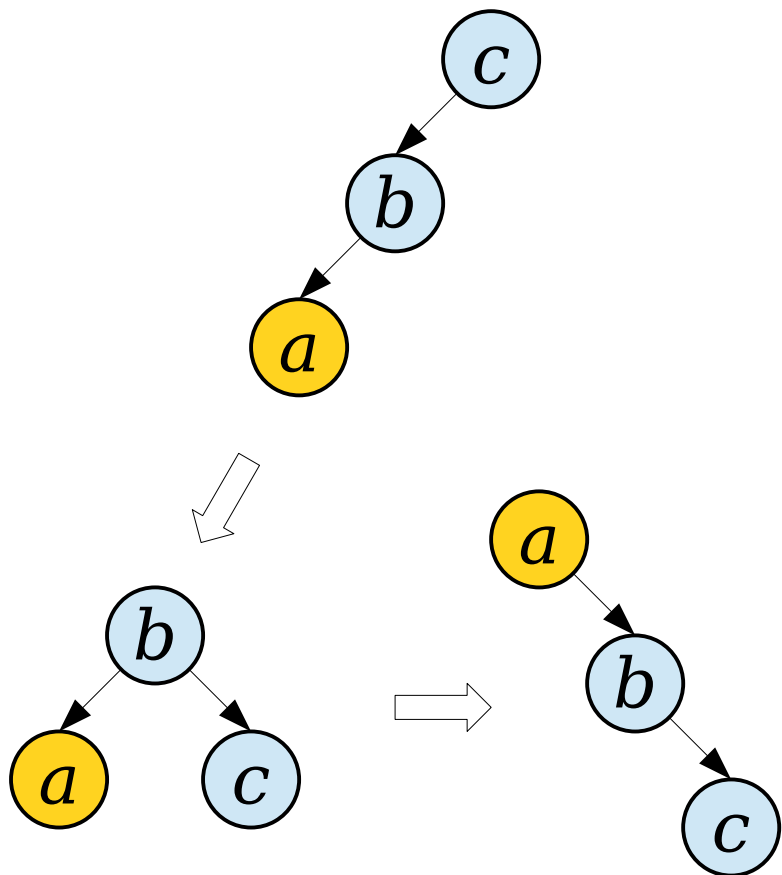
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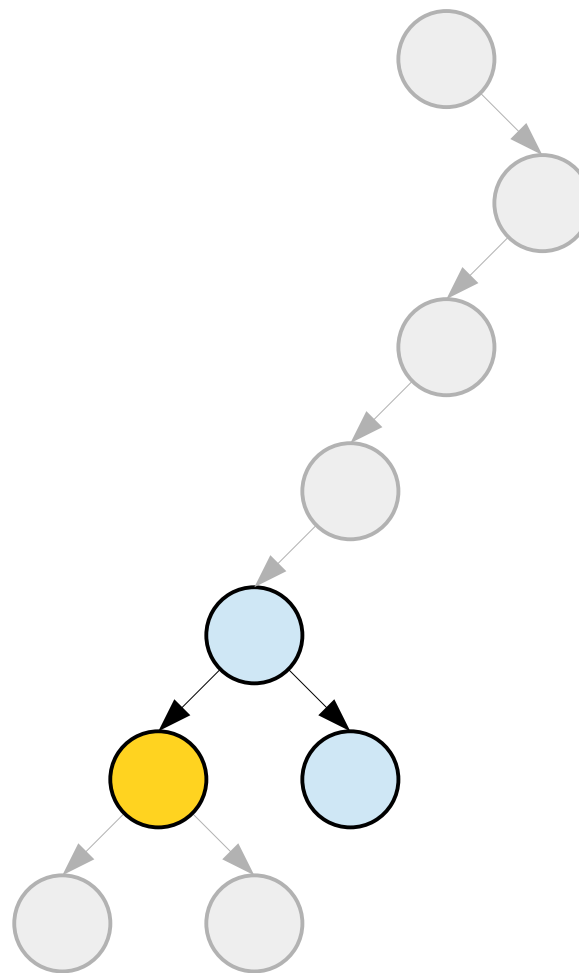
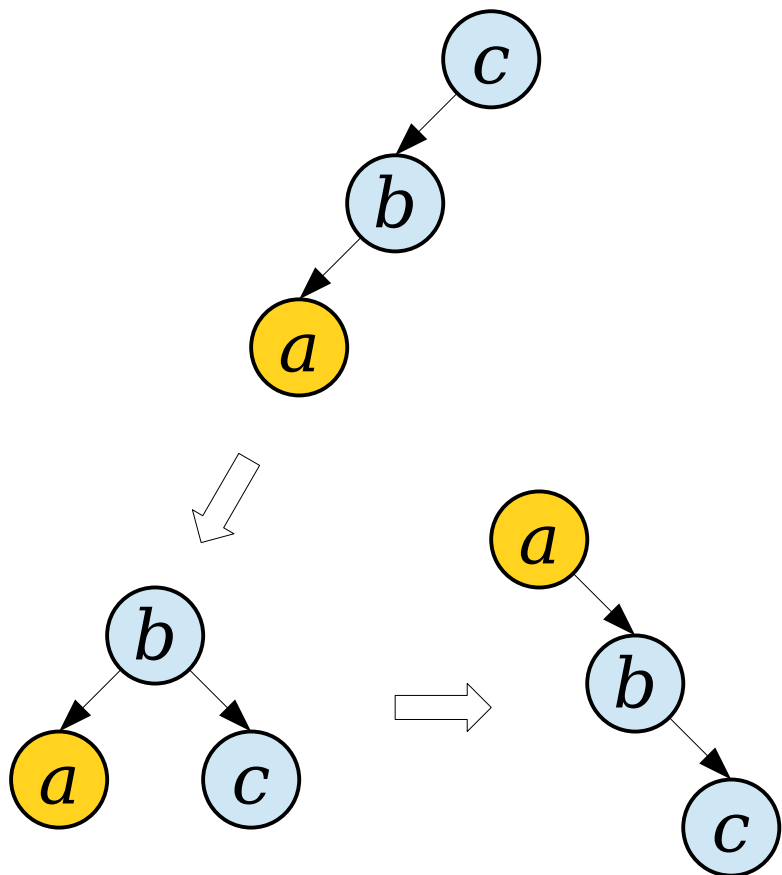
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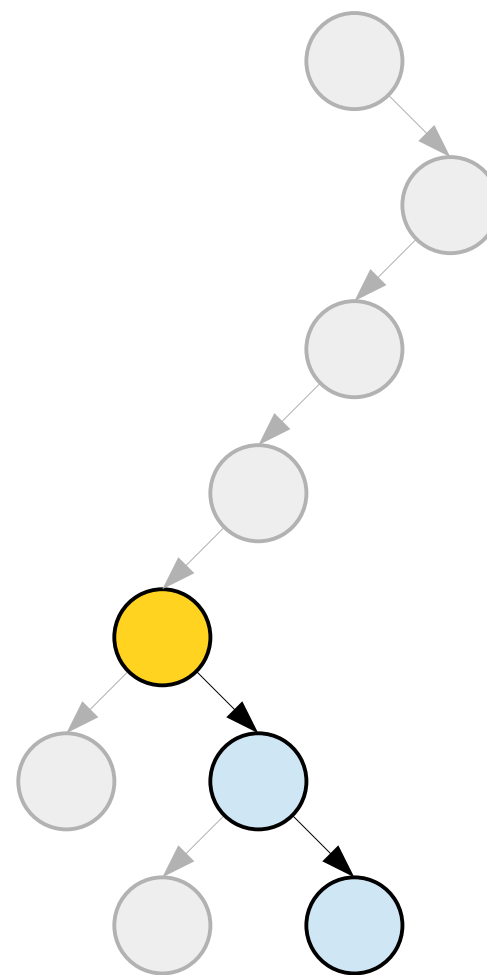
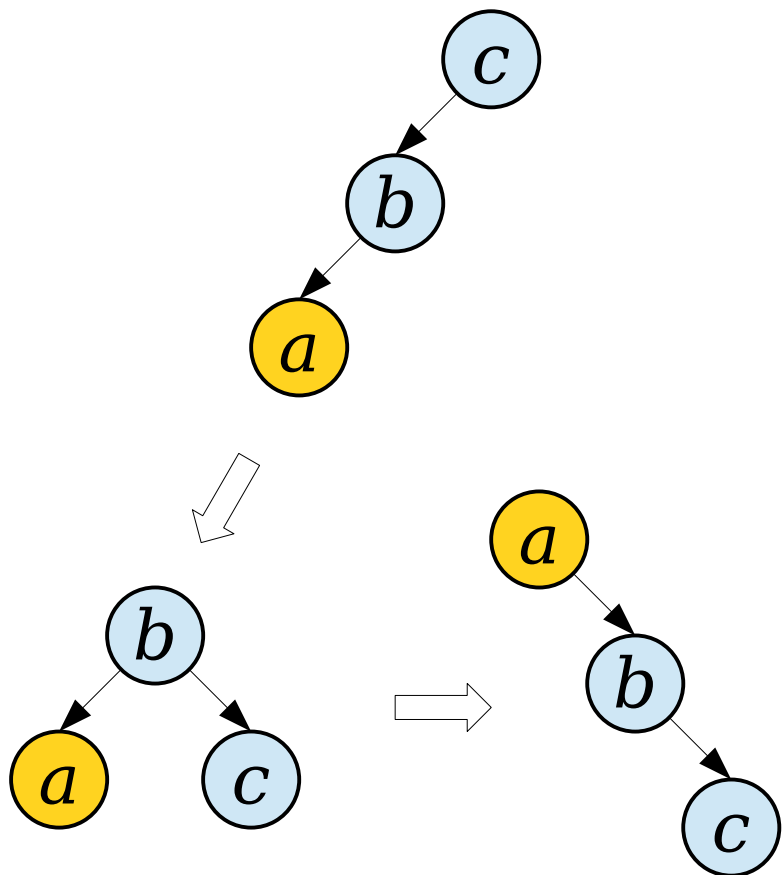
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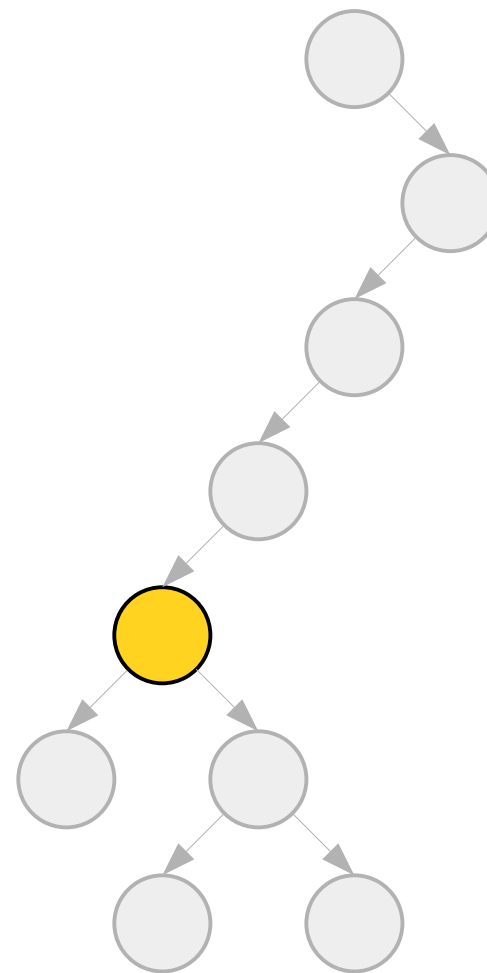
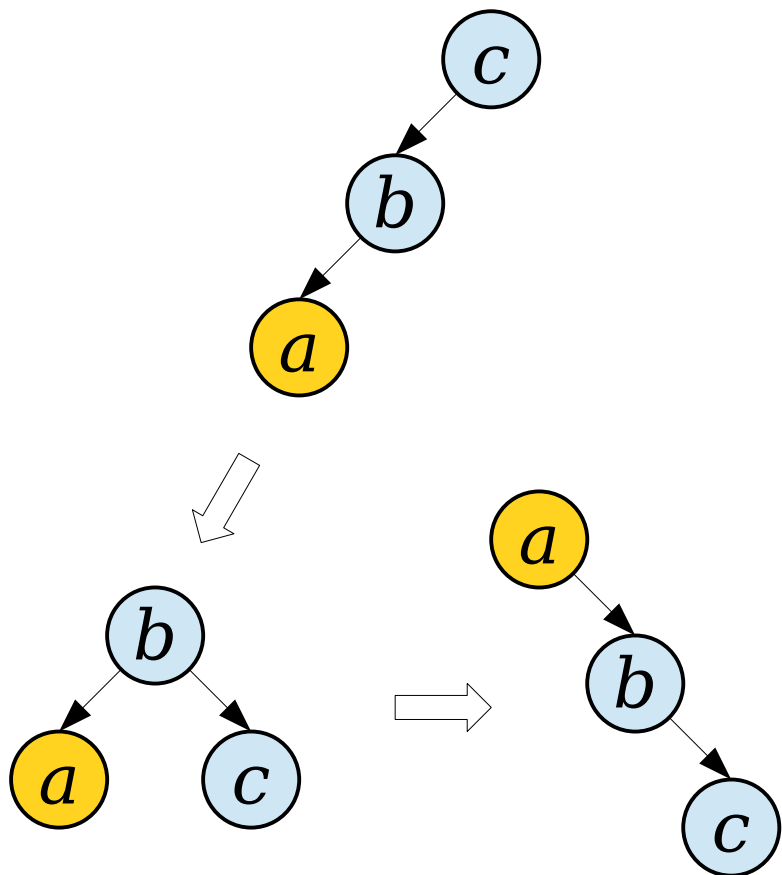
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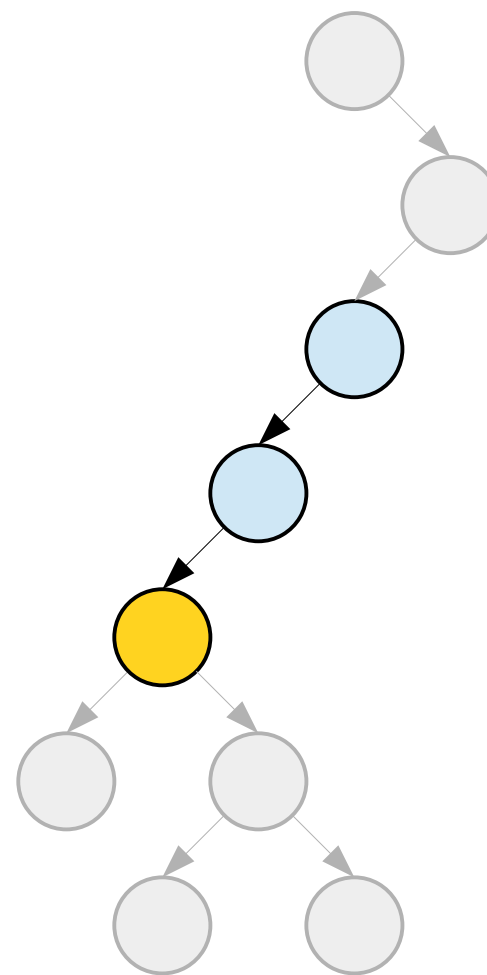
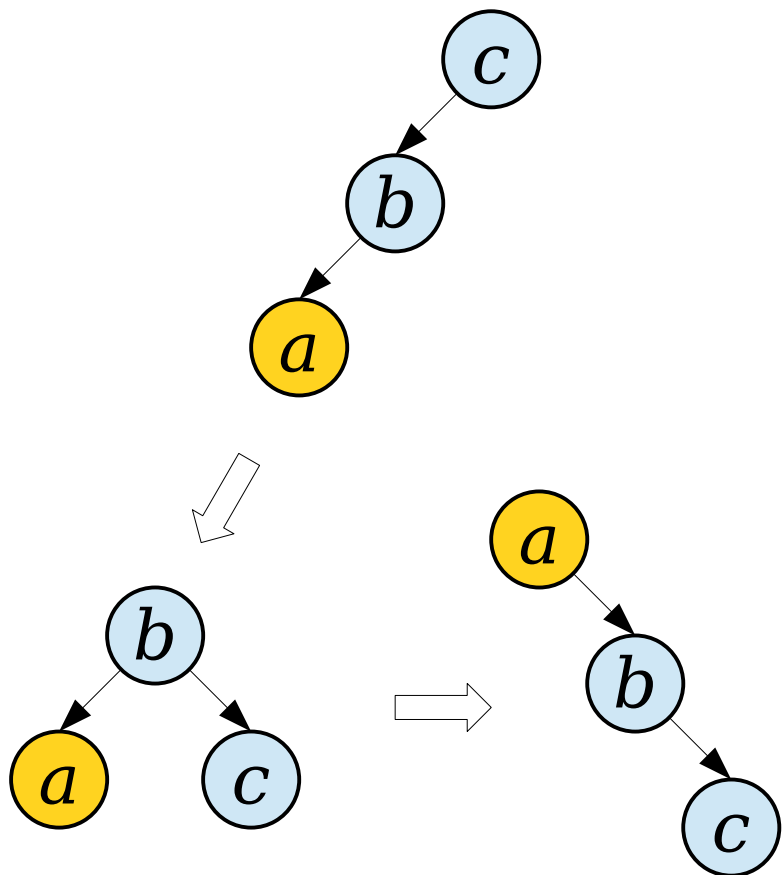


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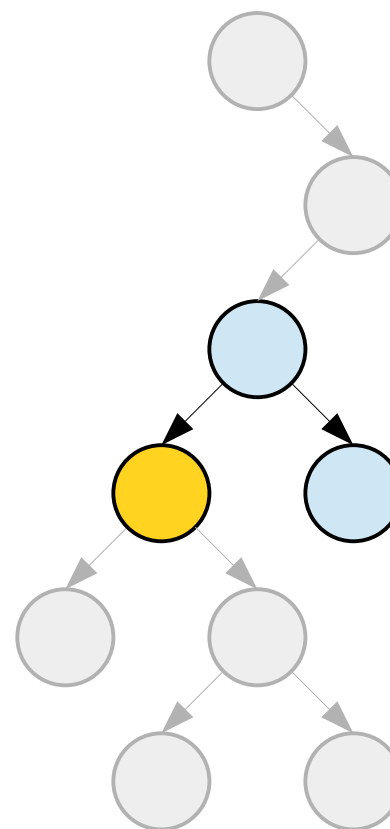
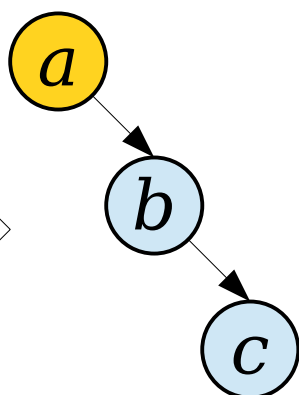
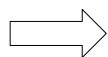
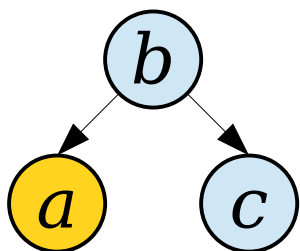
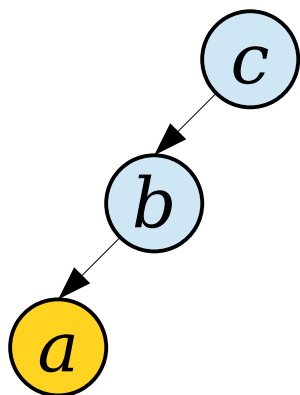
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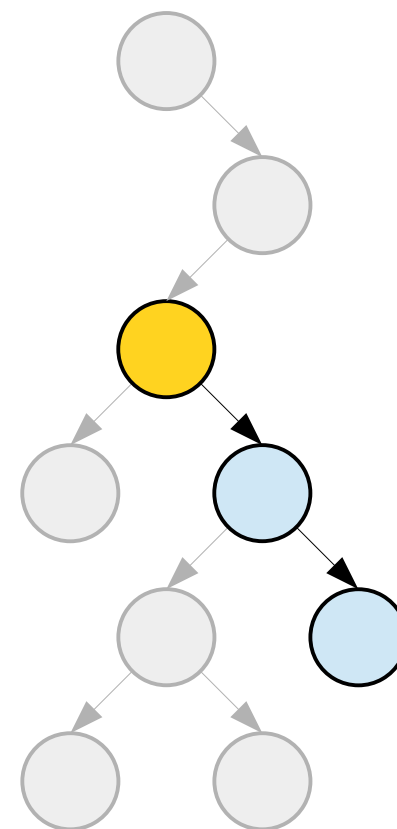
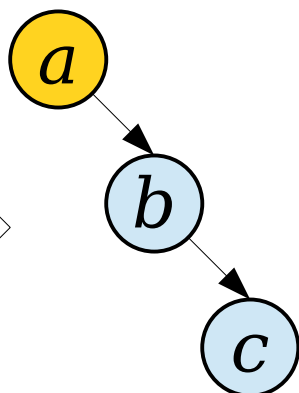
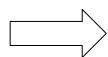
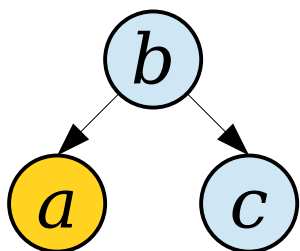
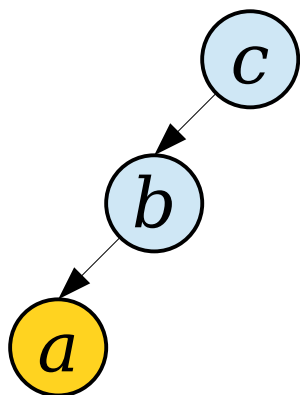
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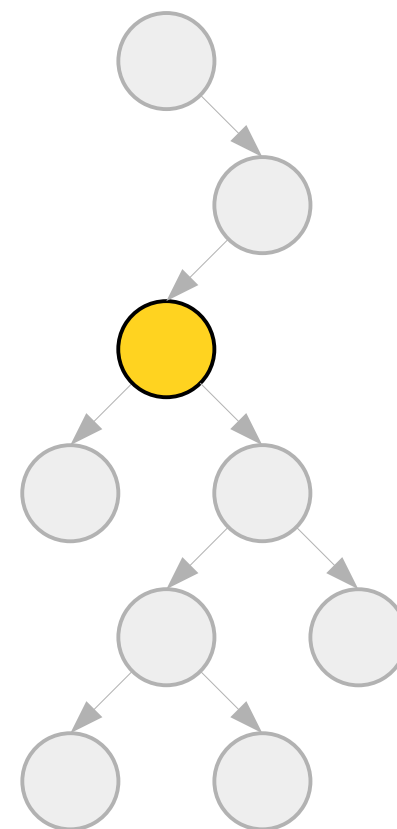
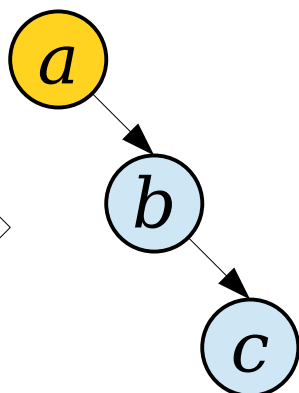
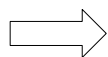
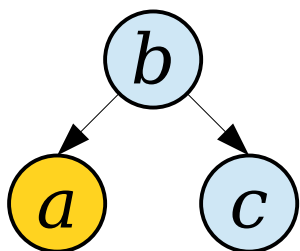
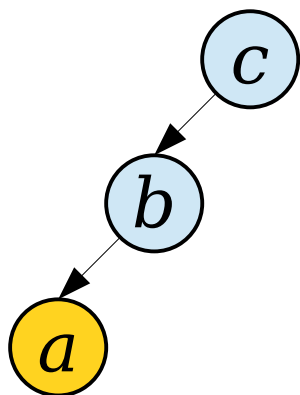
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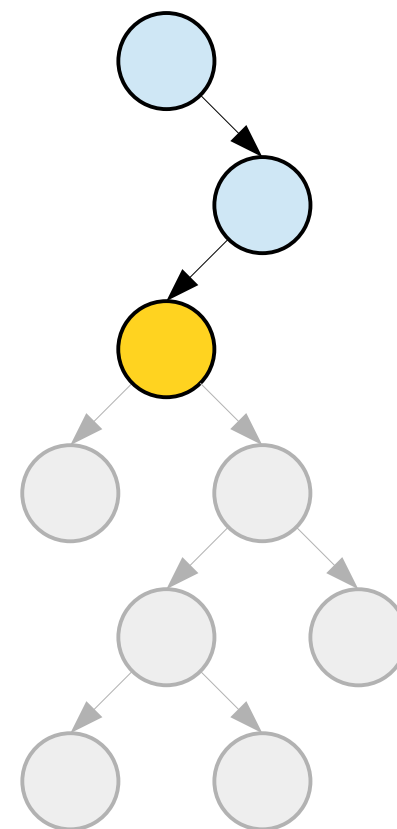
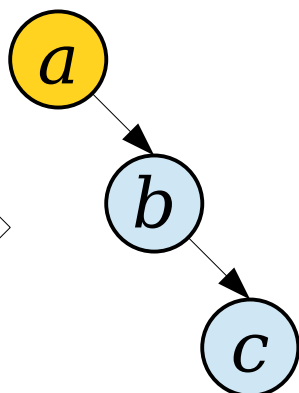
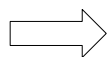
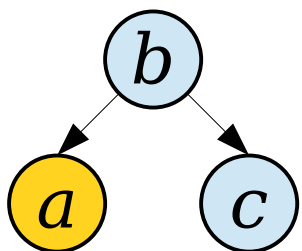
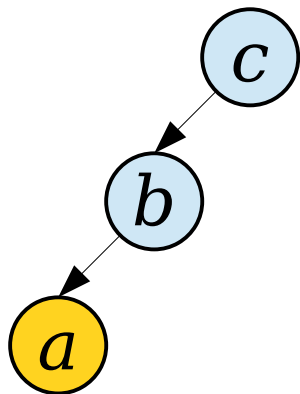
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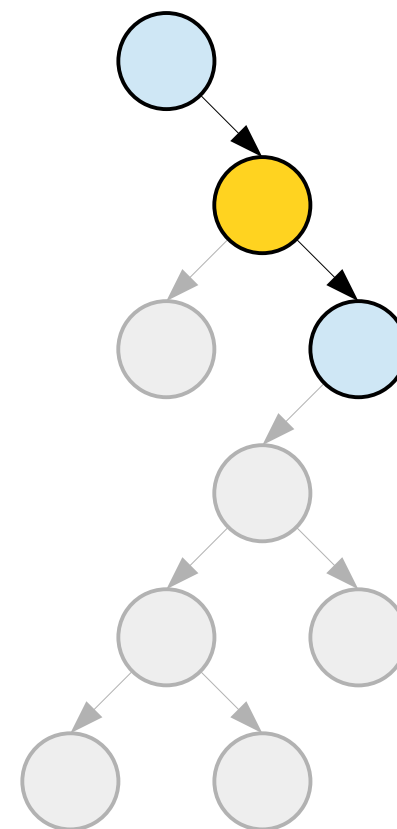
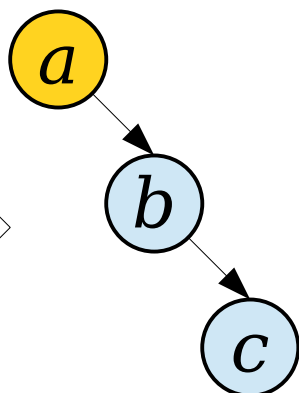
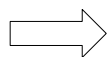
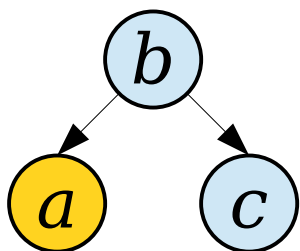
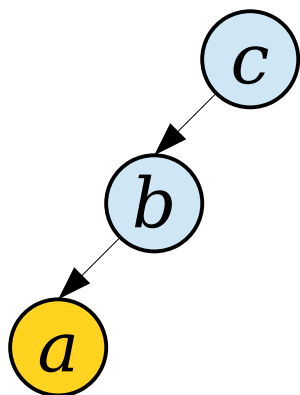
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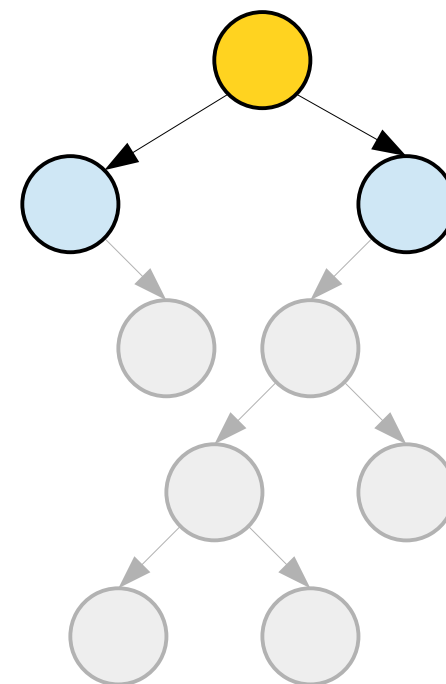
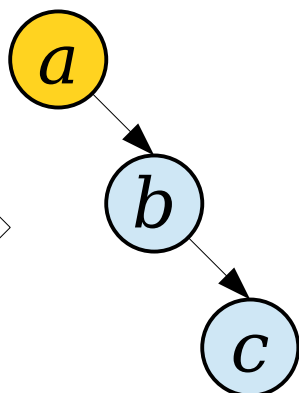
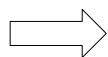
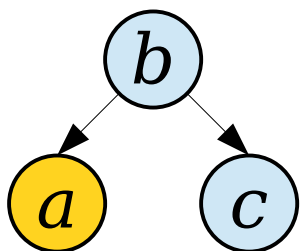
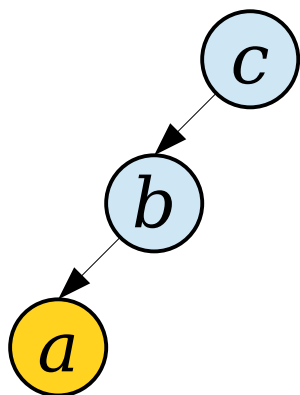
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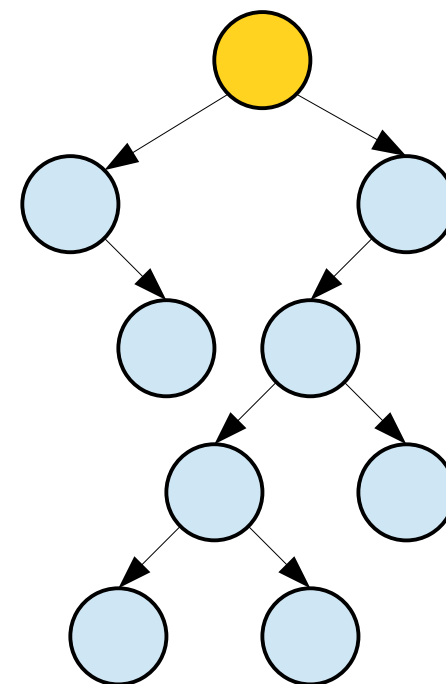
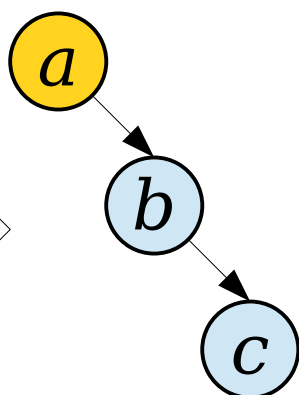
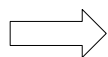
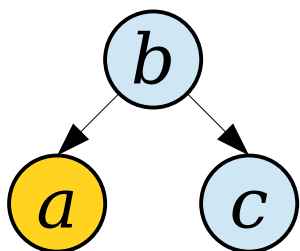
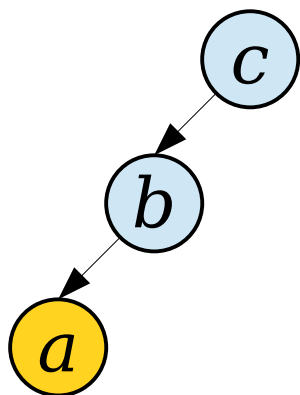
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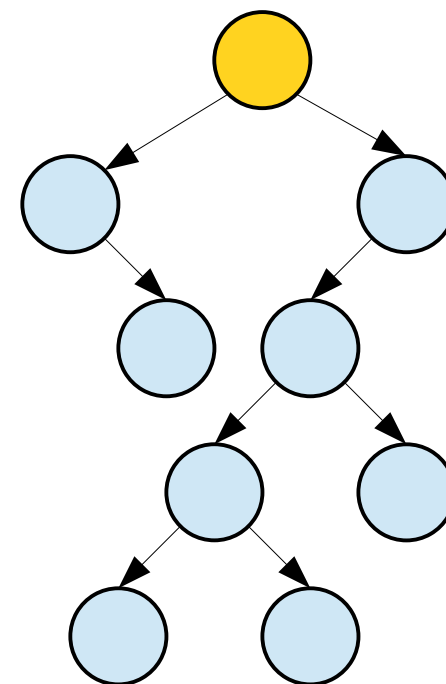
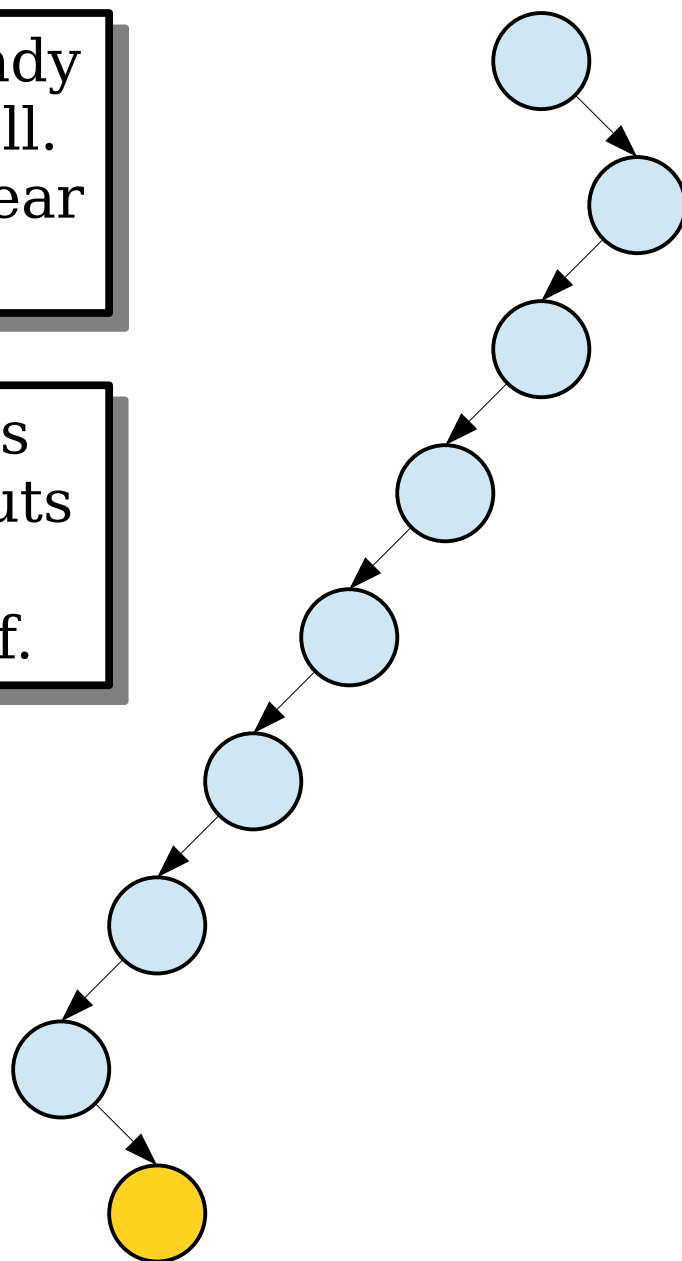
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**Observation:** This new rule roughly cuts the height of the access path in half.



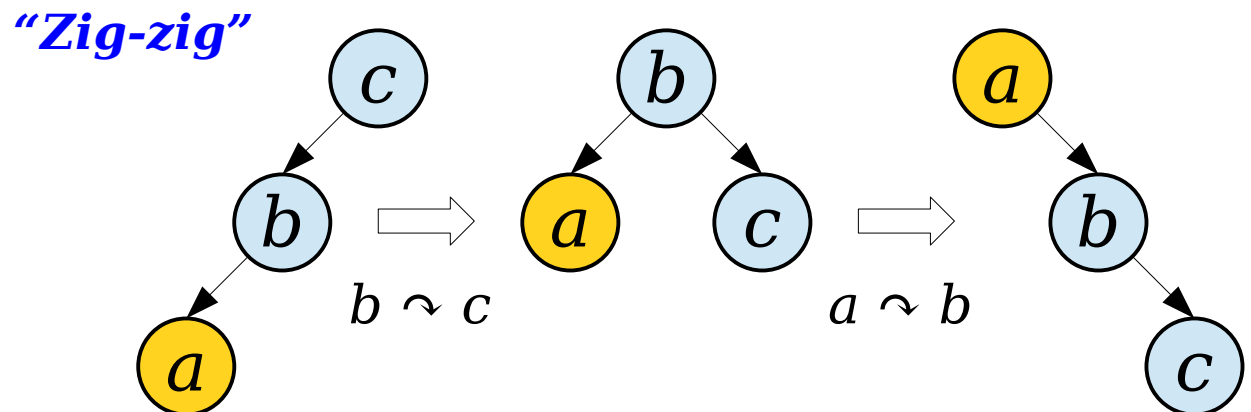
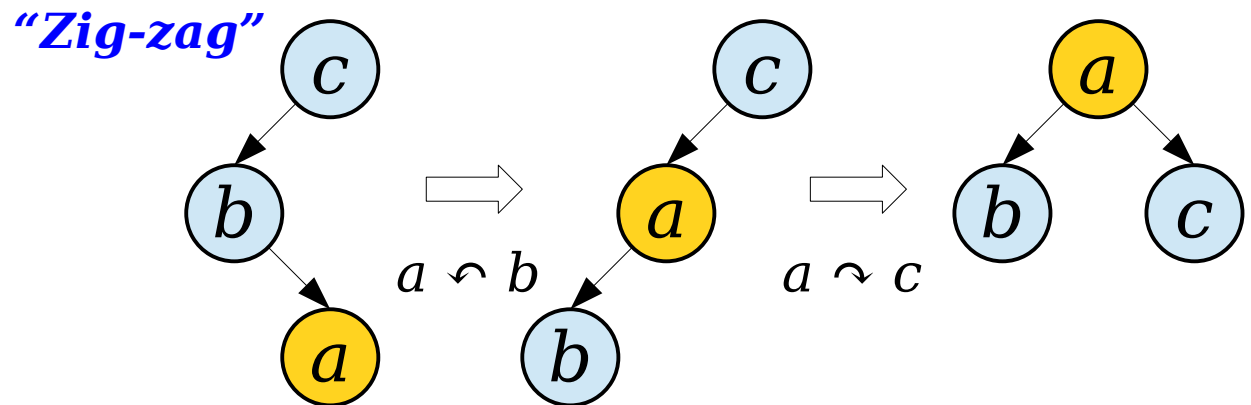
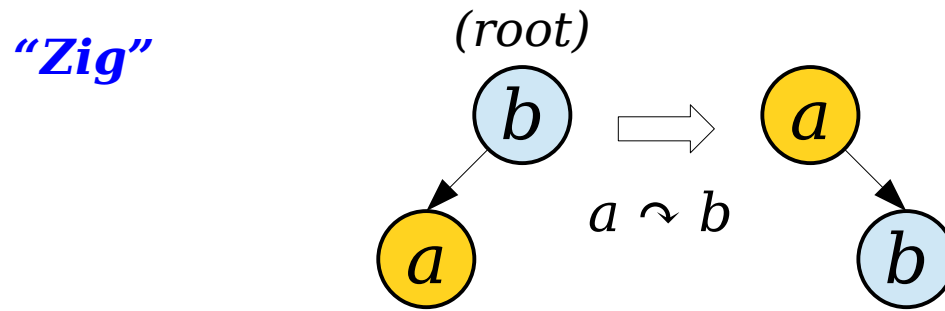
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**Question:** How do we fix rotate-to-root to work well with long chains of nodes?

This procedure for moving a node to the root of the tree is called **splaying**.

**Intuition:** Use rotate-to-root, except when nodes chain in the same direction.

**Mechanics:** Look back two steps in the tree and apply the appropriate rotation rules.



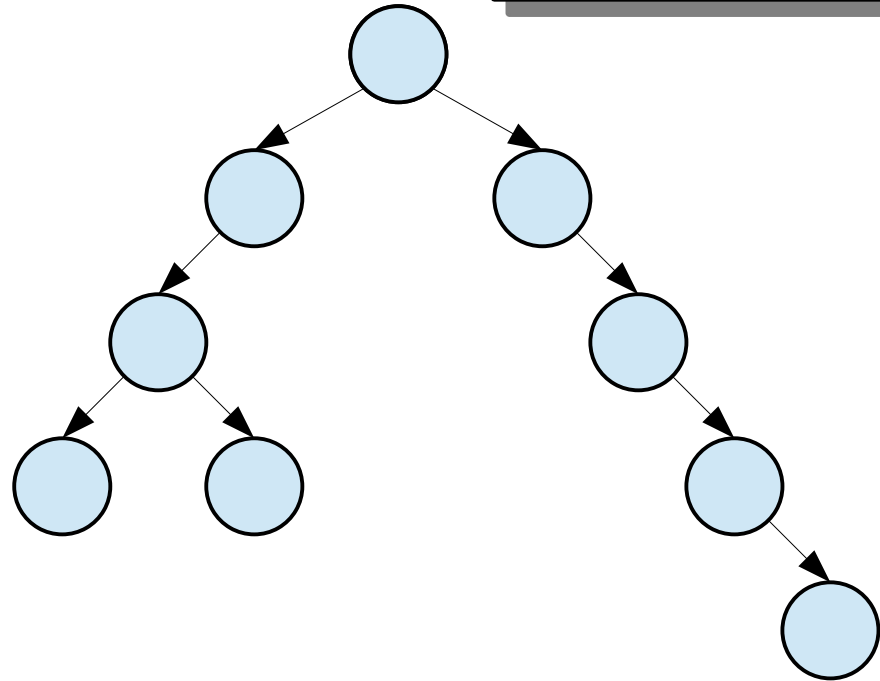
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A **splay tree** is a regular BST where we splay the last node touched after each operation.

**Theorem:** The amortized cost of splaying a node is  $O(\log n)$ .

**Claim:** Every splay tree operation cost is bounded by  $O(1)$  splays and takes amortized time  $O(\log n)$ .

**Insert:** Add as usual, then splay the new node.



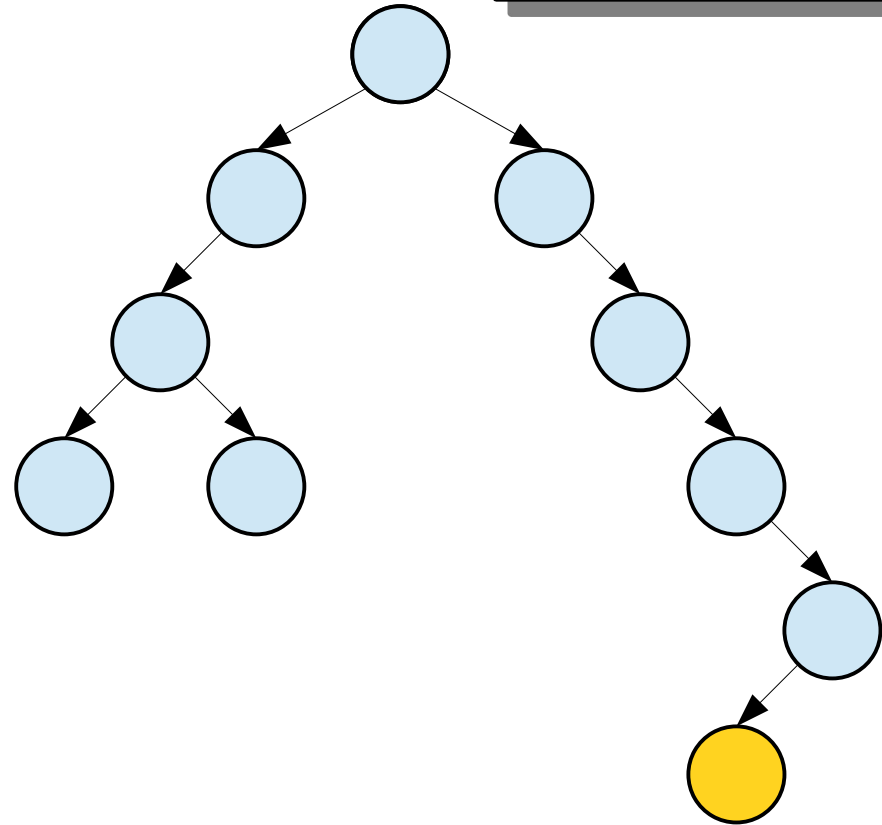
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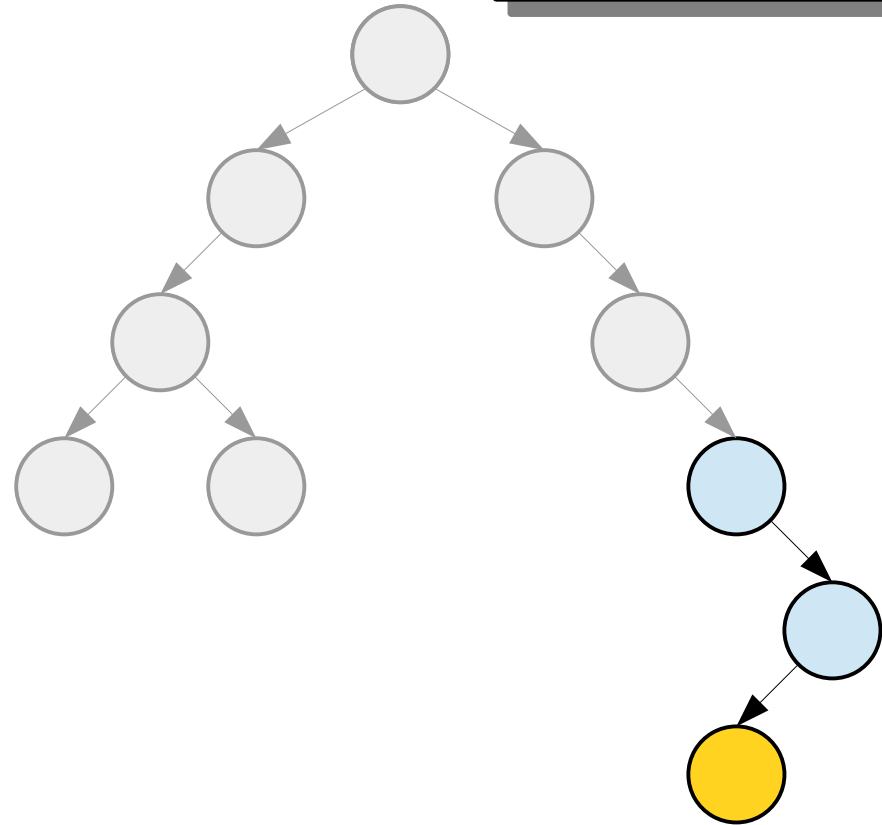
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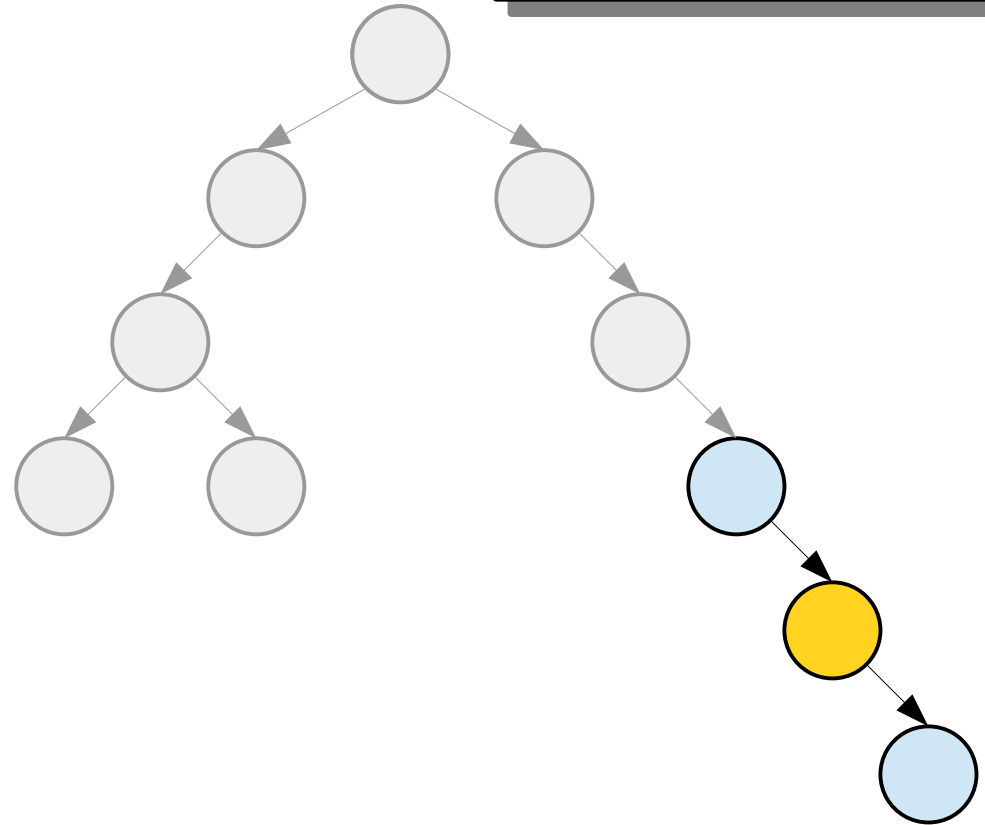
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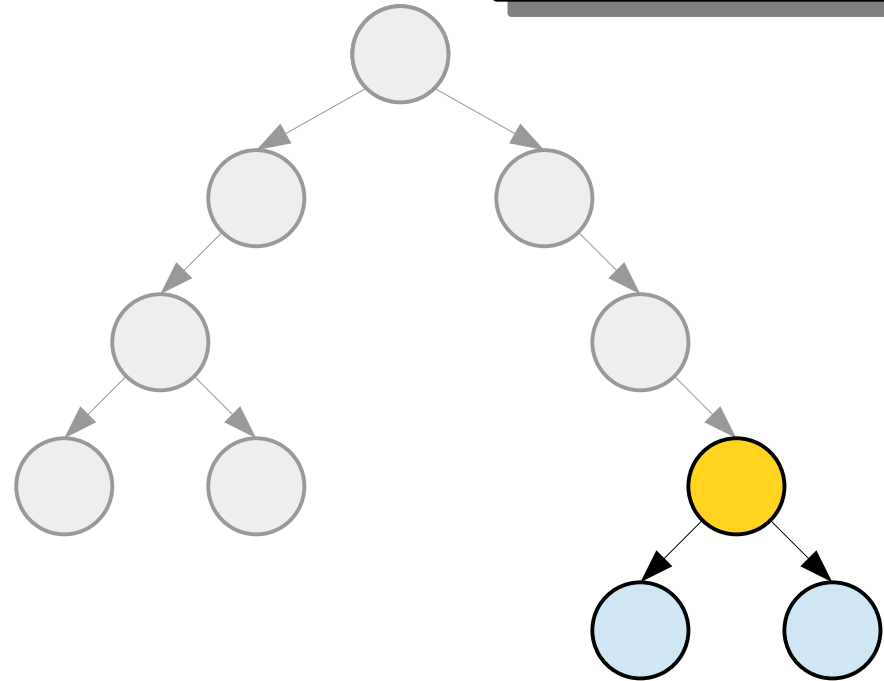
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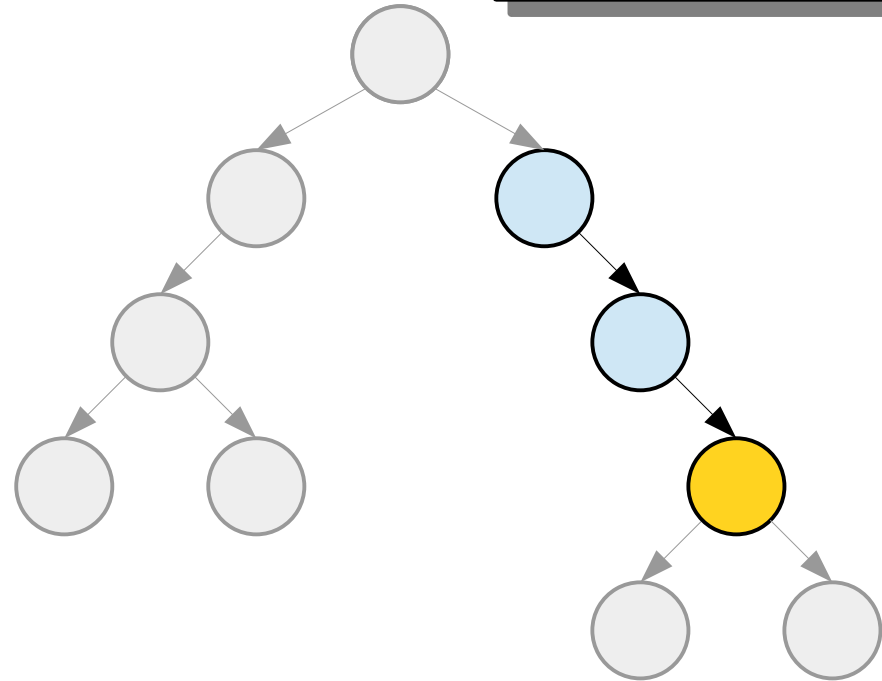


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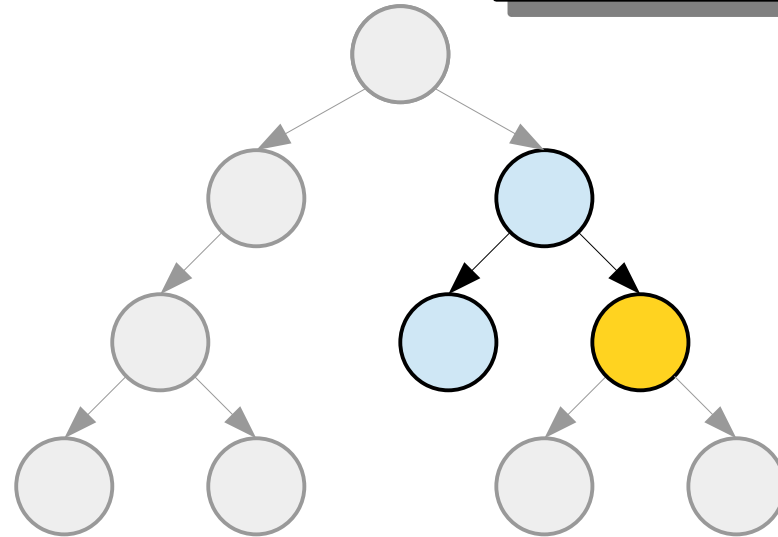
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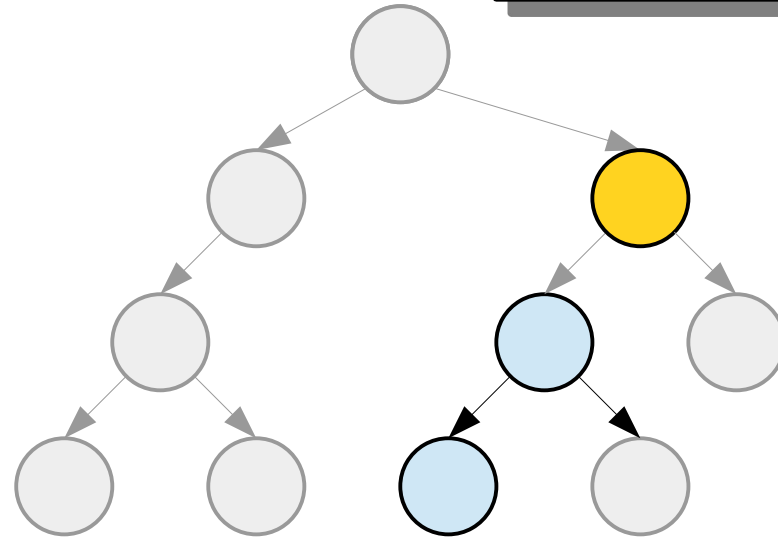
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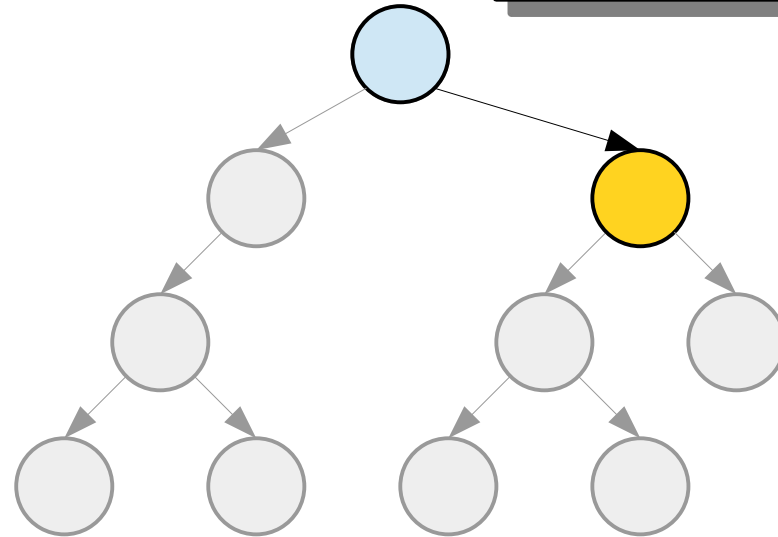
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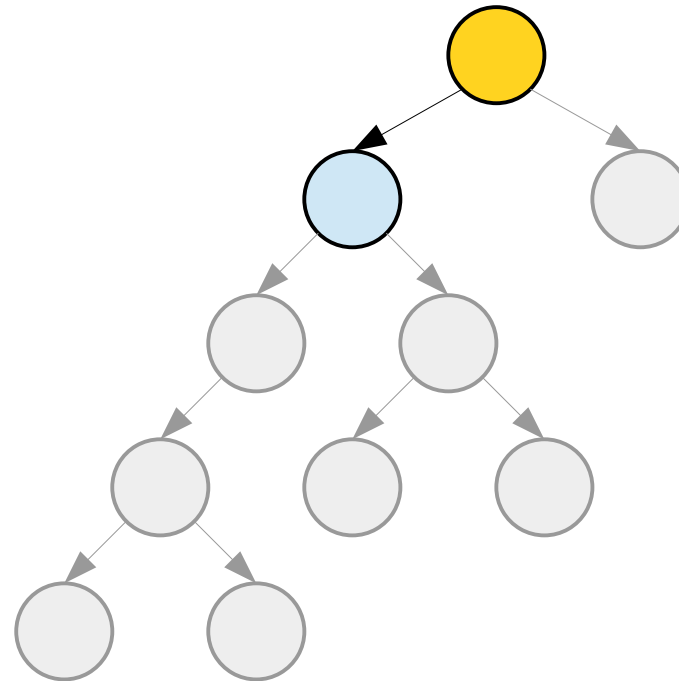
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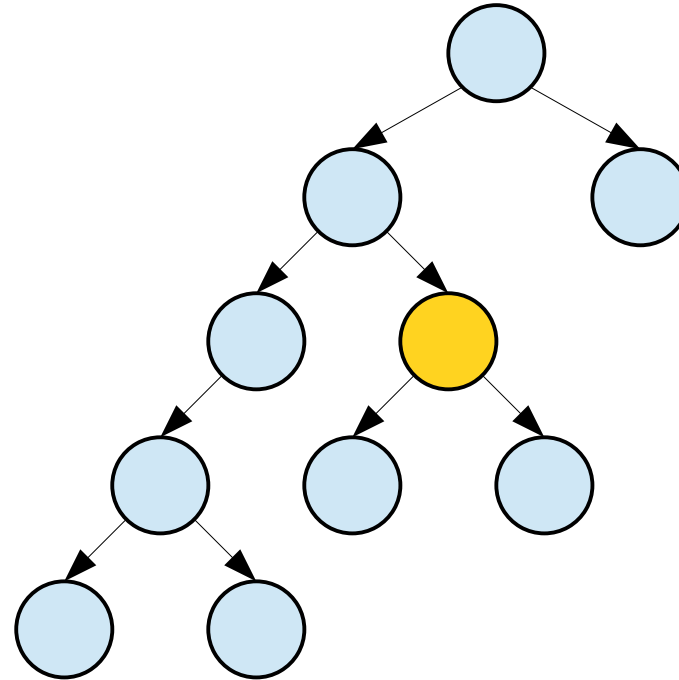


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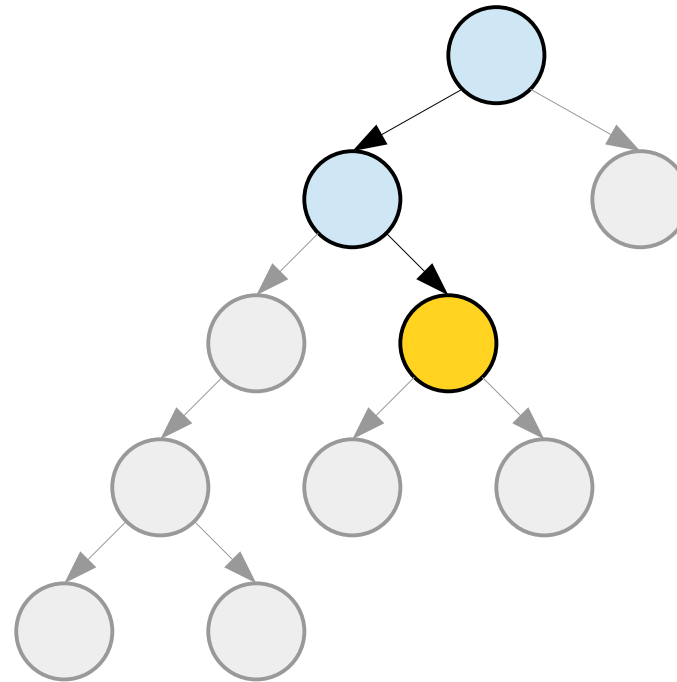


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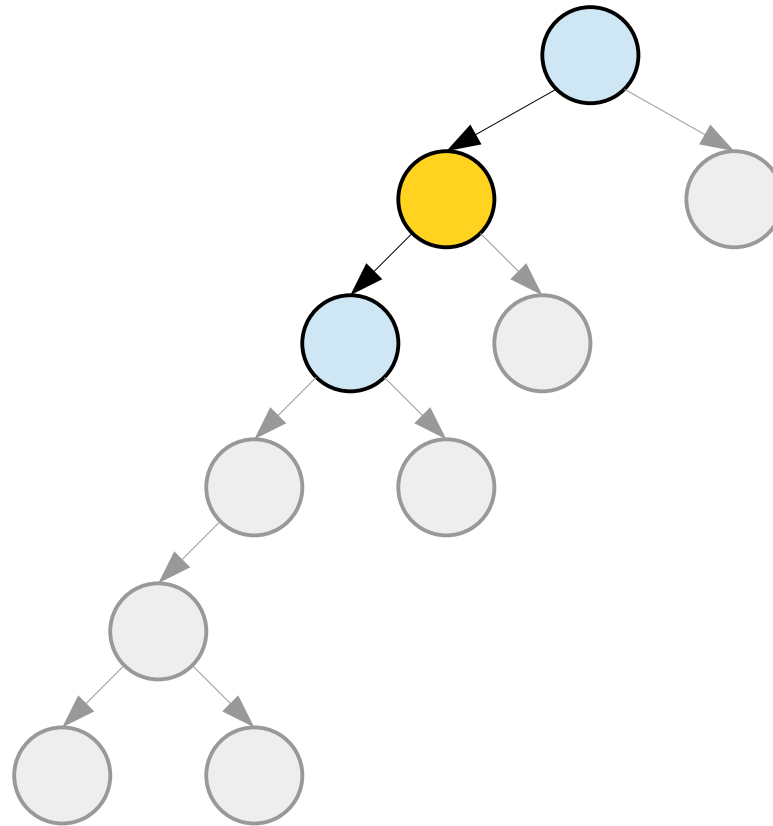
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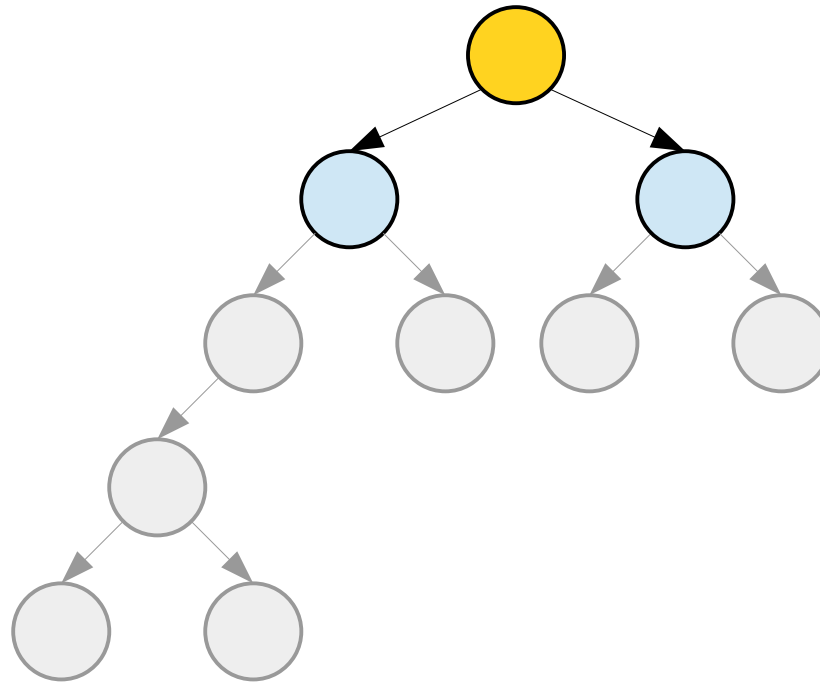
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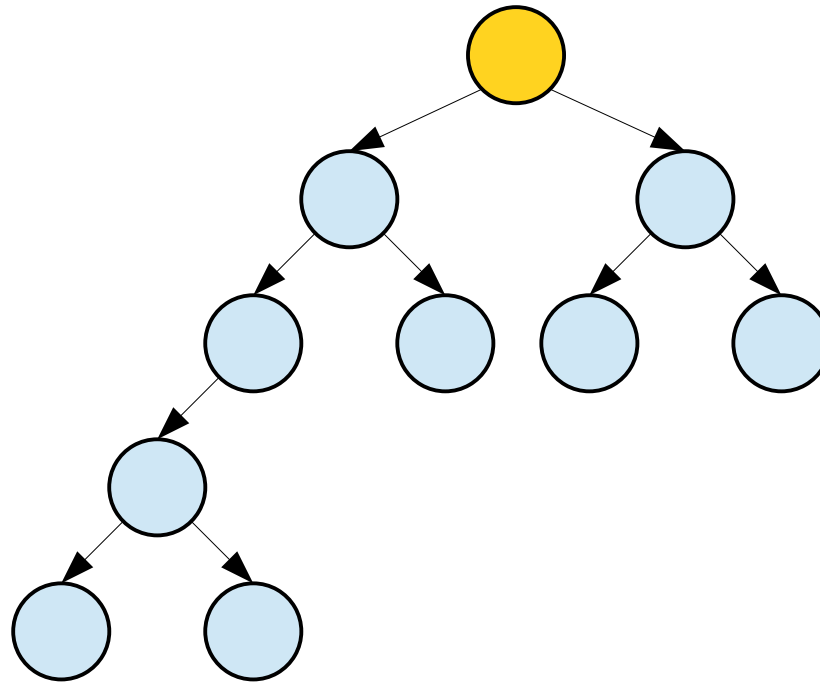
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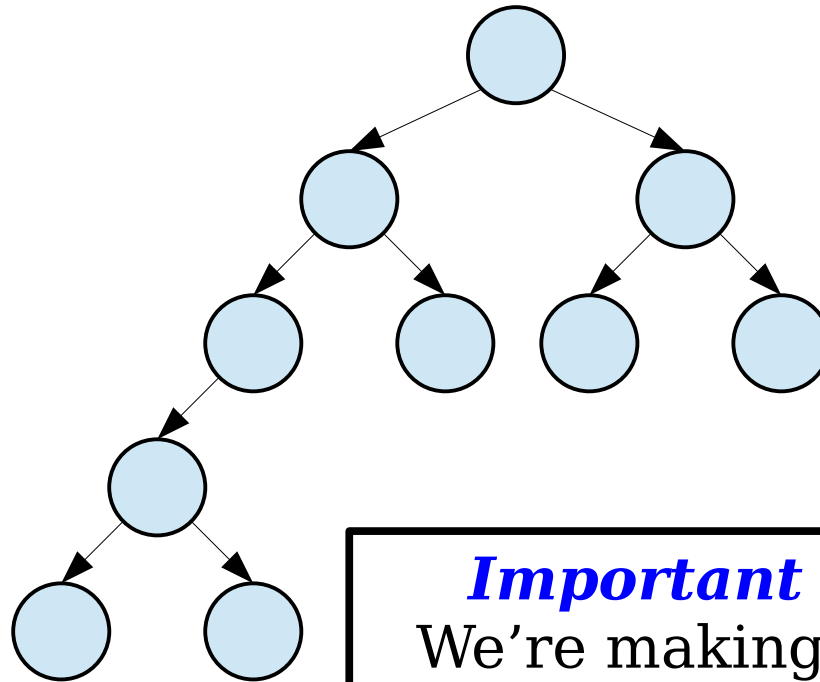
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**Important Fact 1:** We're making no effort whatsoever to keep the tree balanced. The tree shape is purely the result of splaying.

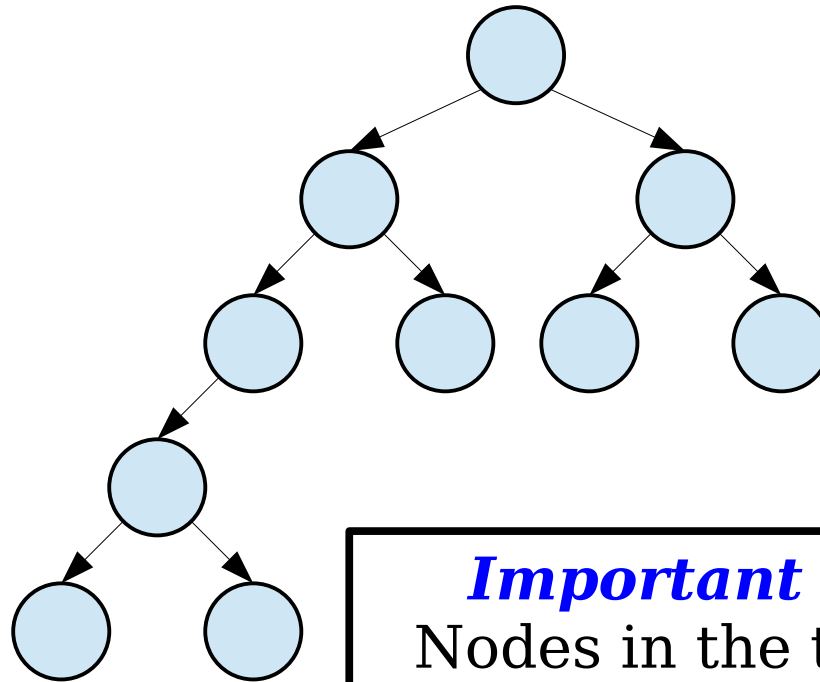
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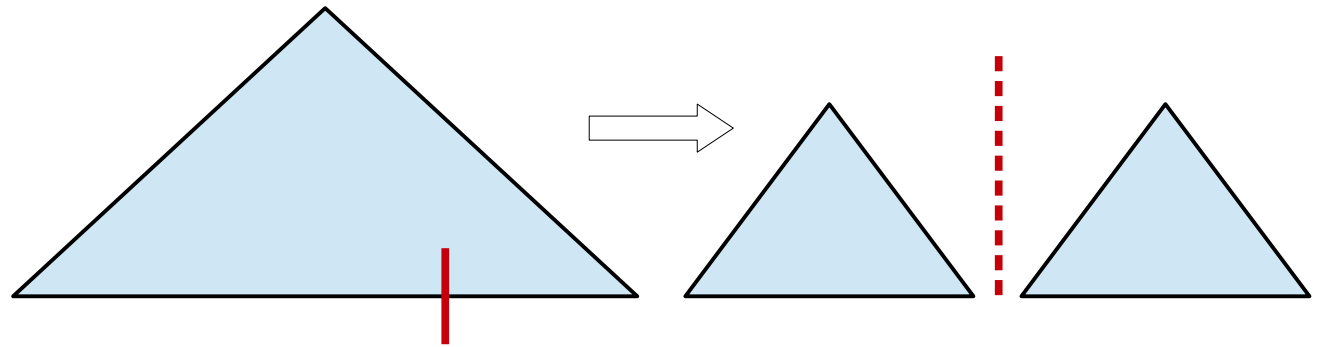
**Important Fact 2:** Nodes in the tree store no extra information beyond left and right pointers. (Contrast with, say, a red/black tree.)

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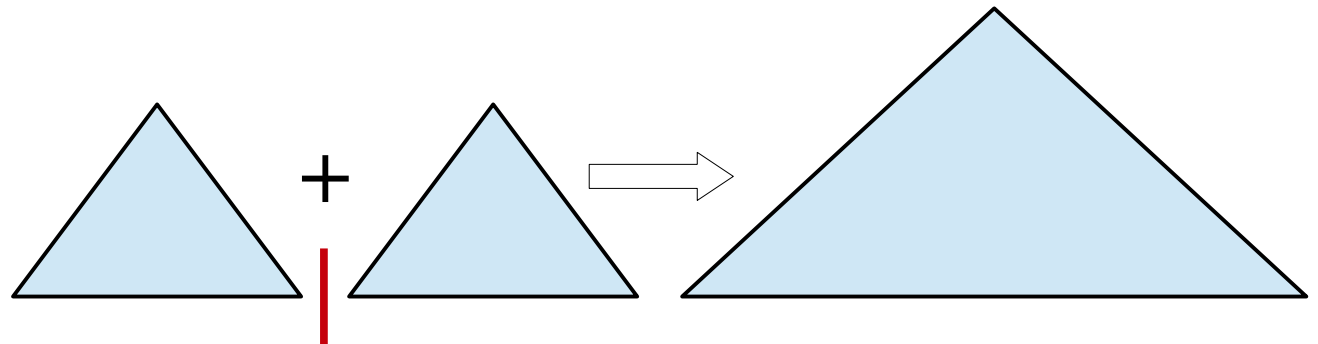
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**Split:** How might you do this?



**Join:** How might you do this?

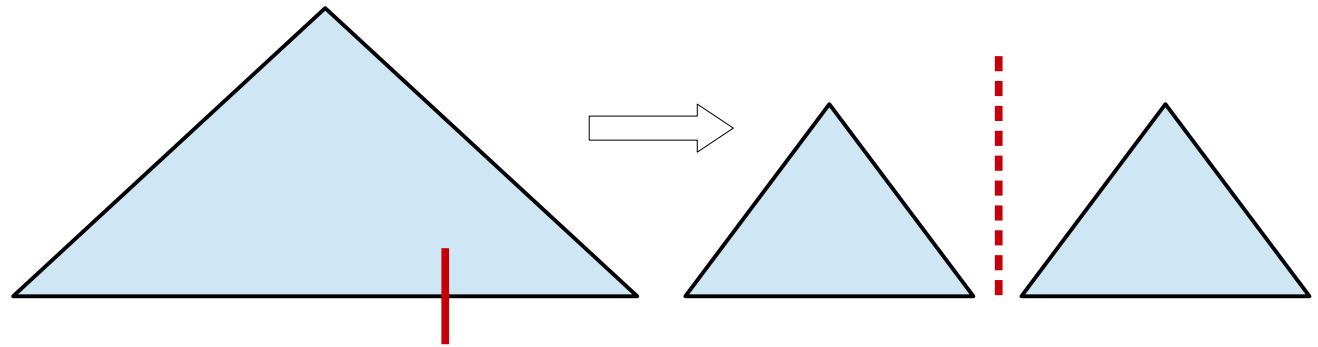
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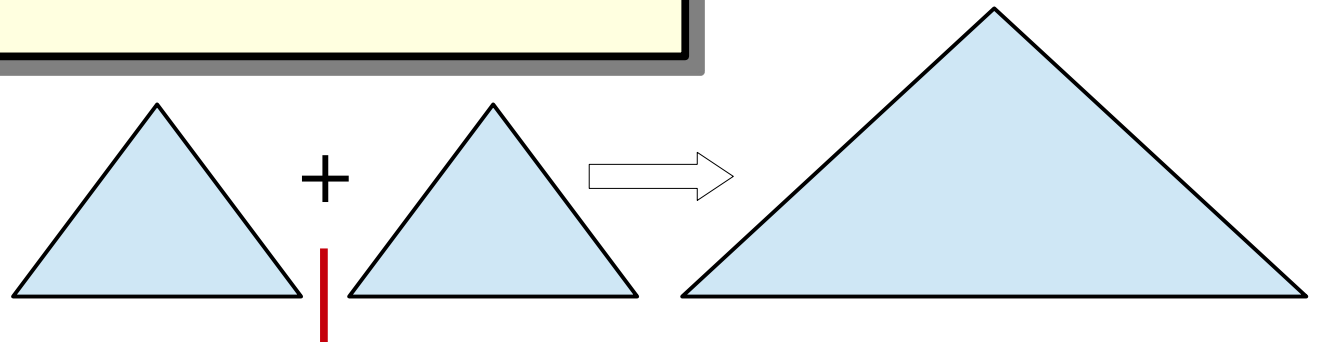
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Answer at <https://pollev.com/cs166spr23>

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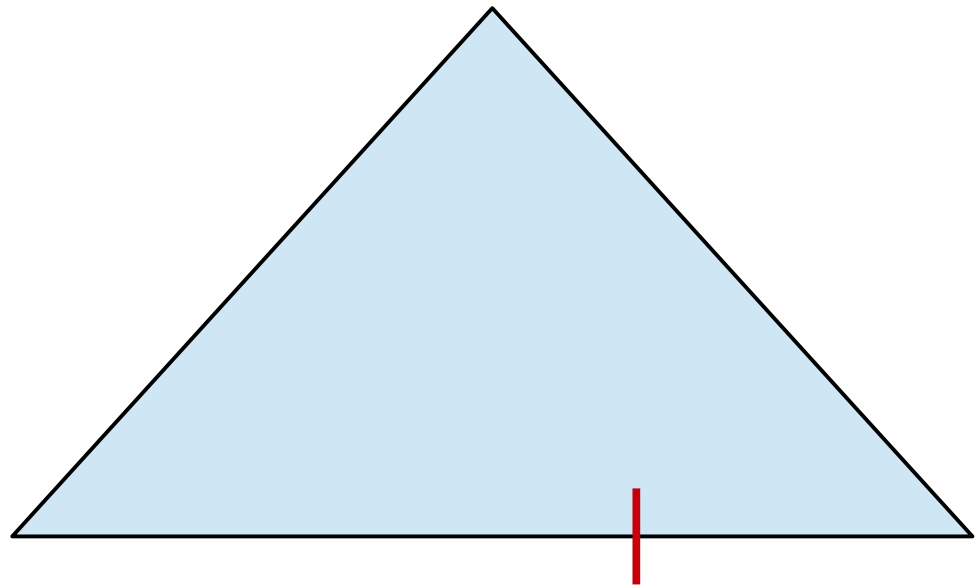


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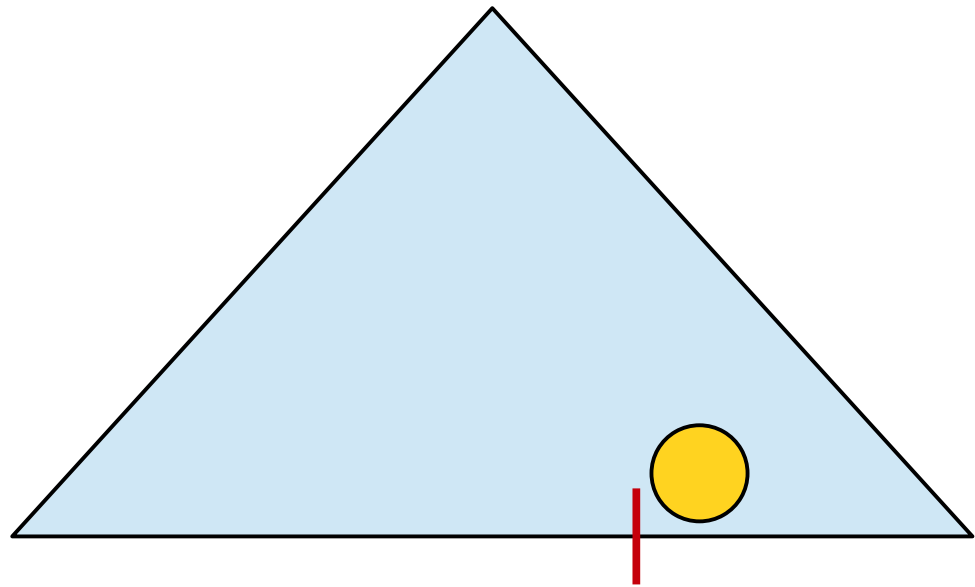
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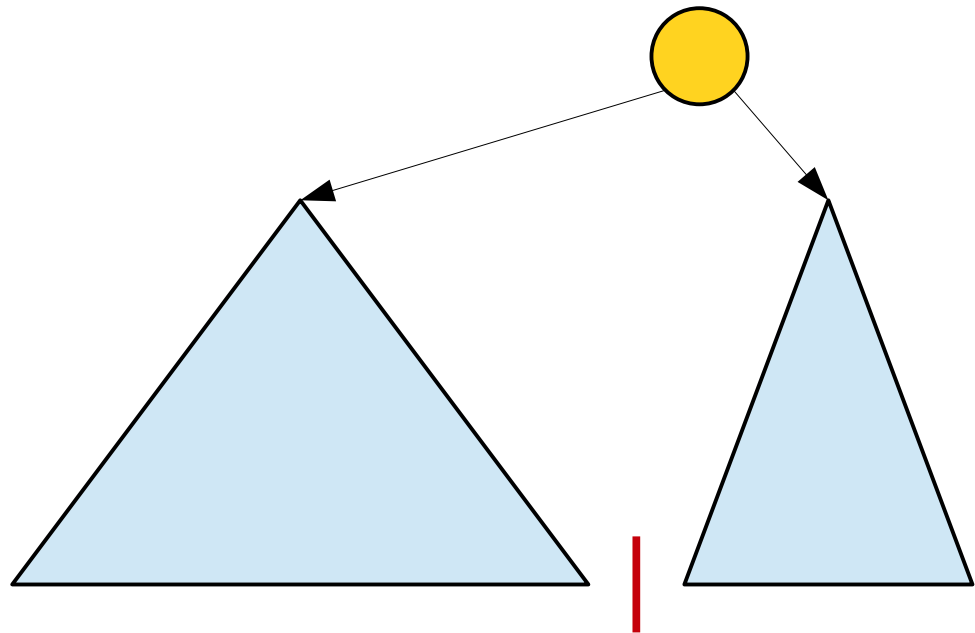
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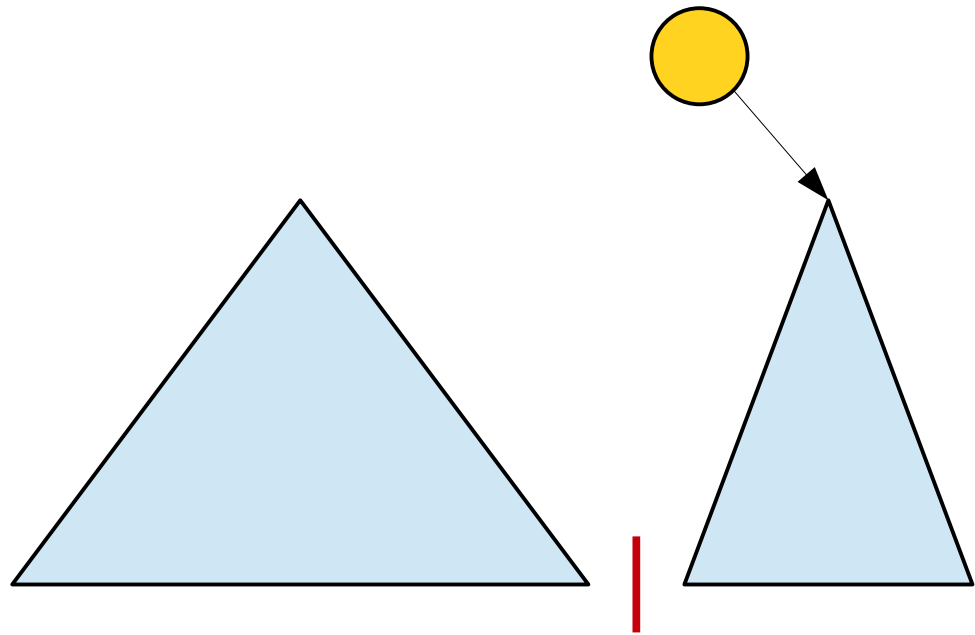
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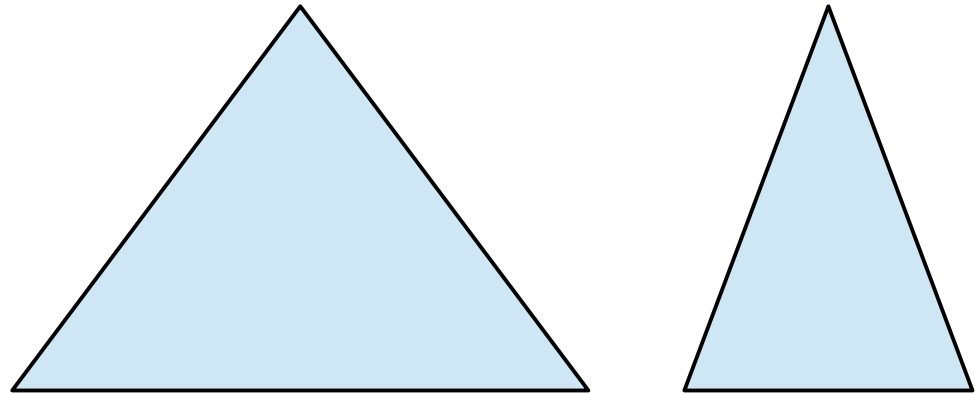
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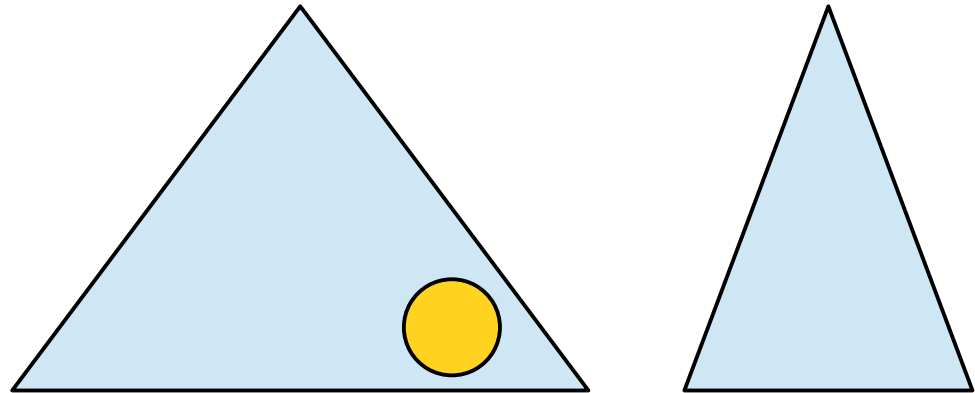
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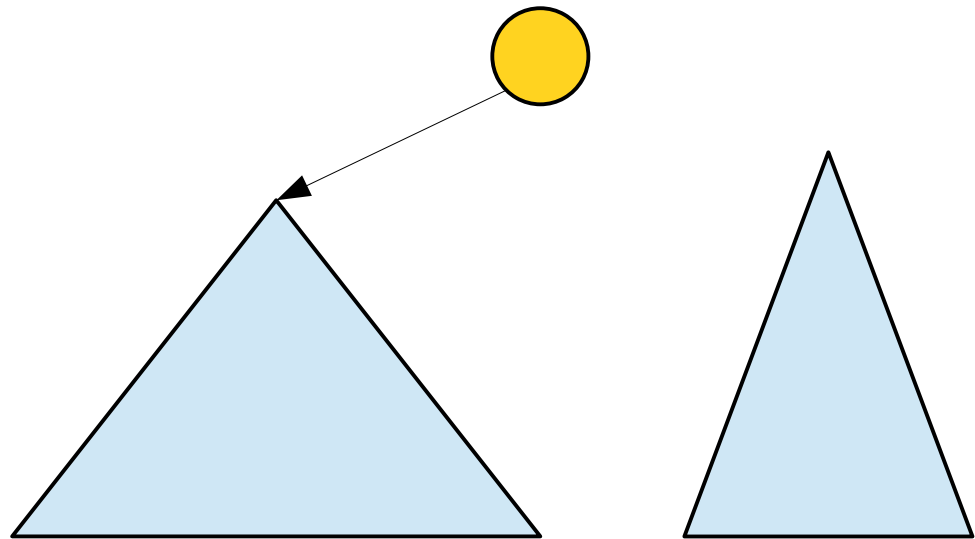
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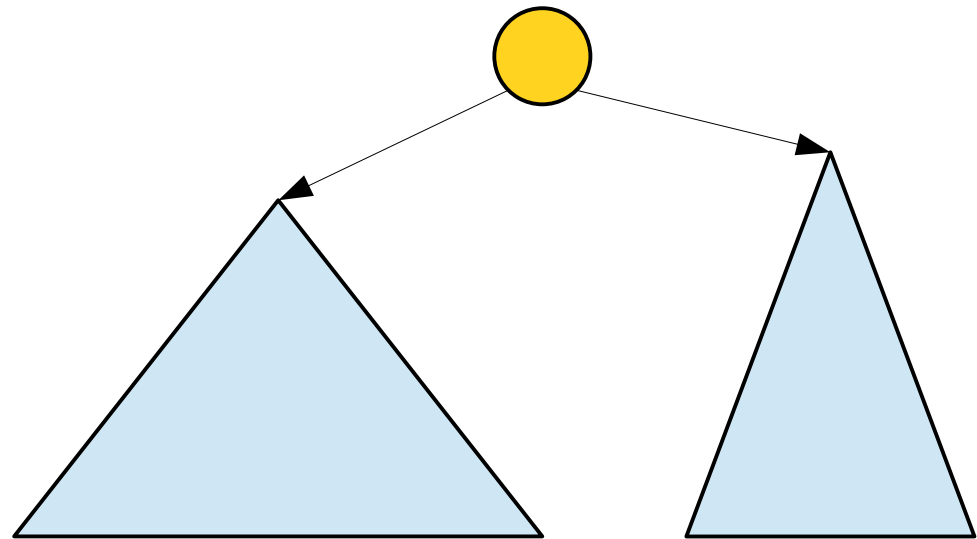
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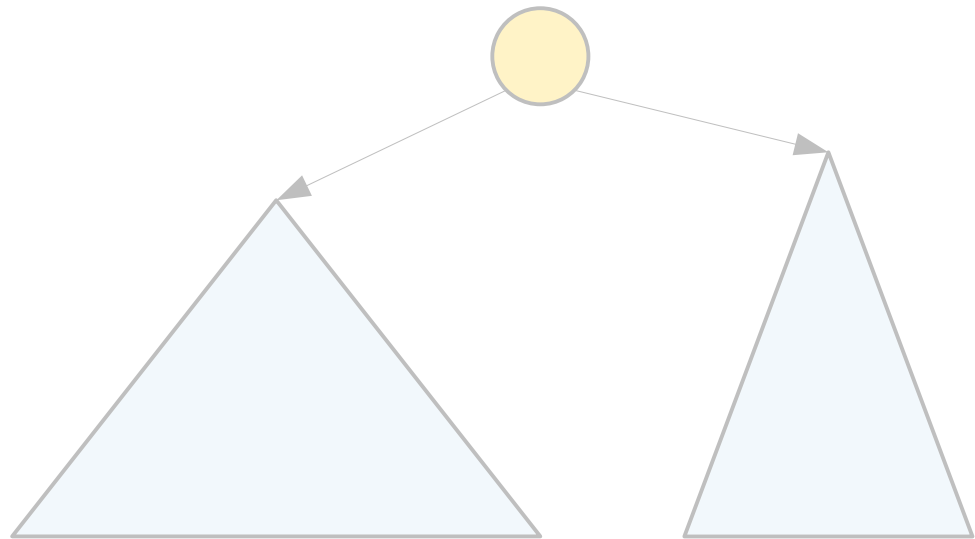


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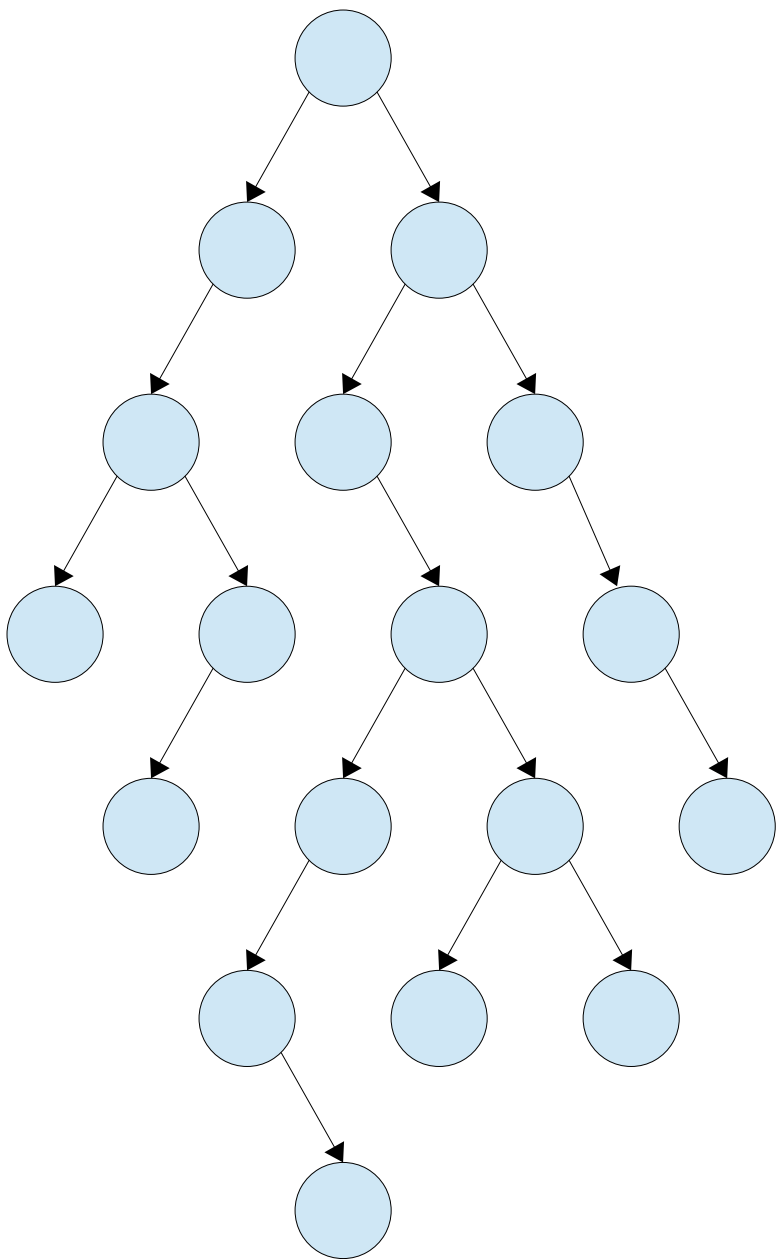
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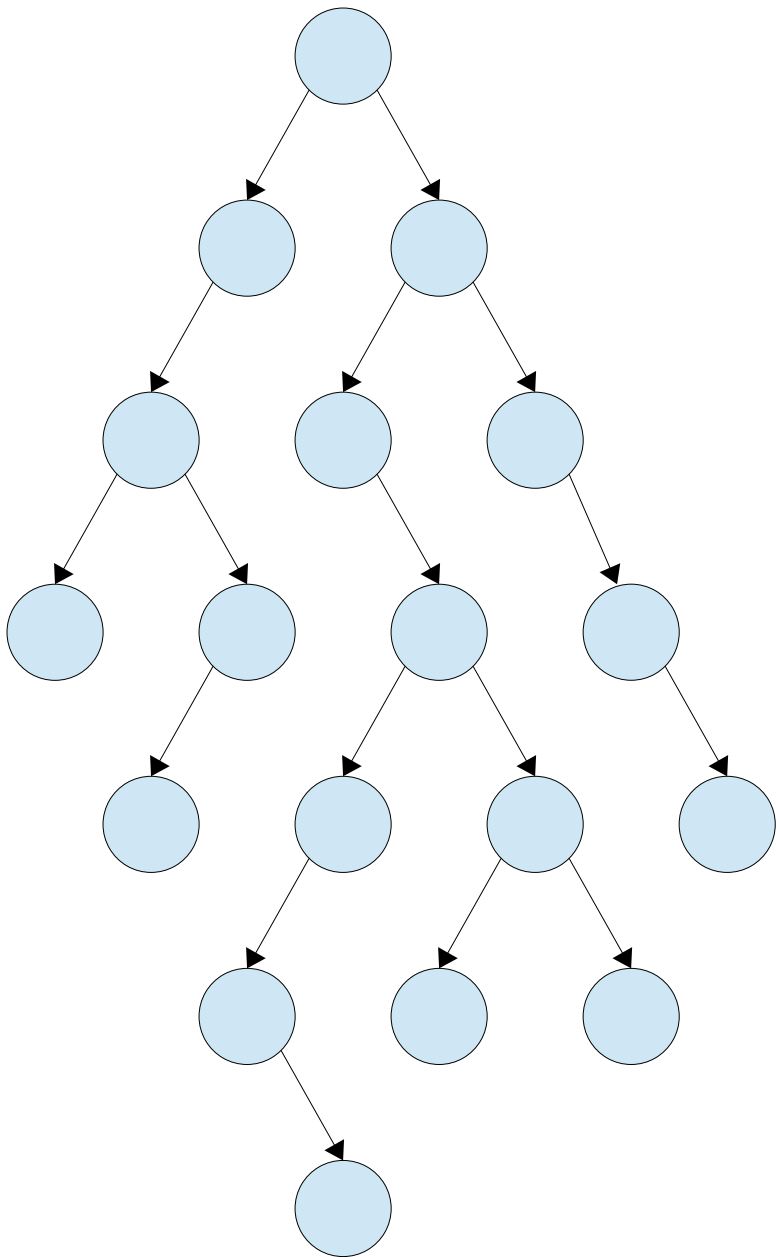
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***Question:*** Why is splaying fast?

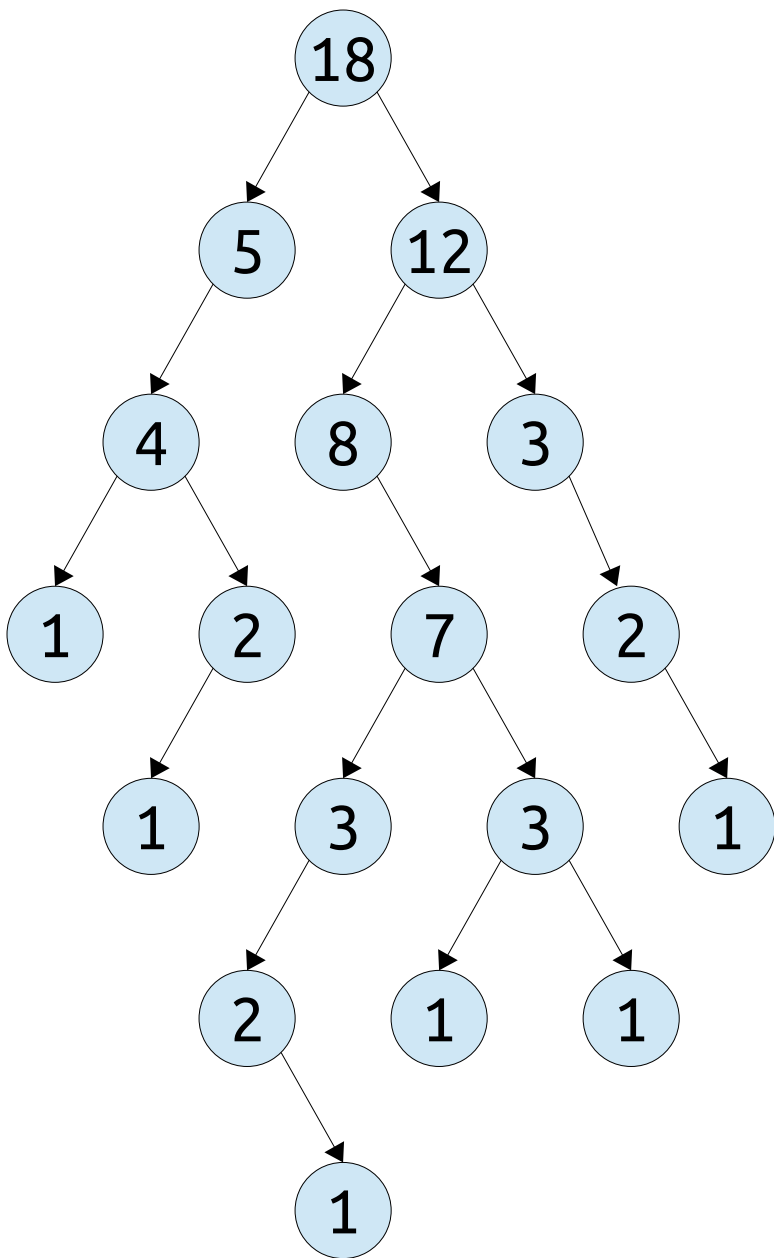
Let  $s(x)$  denote the number of nodes in the subtree rooted at  $x$ .



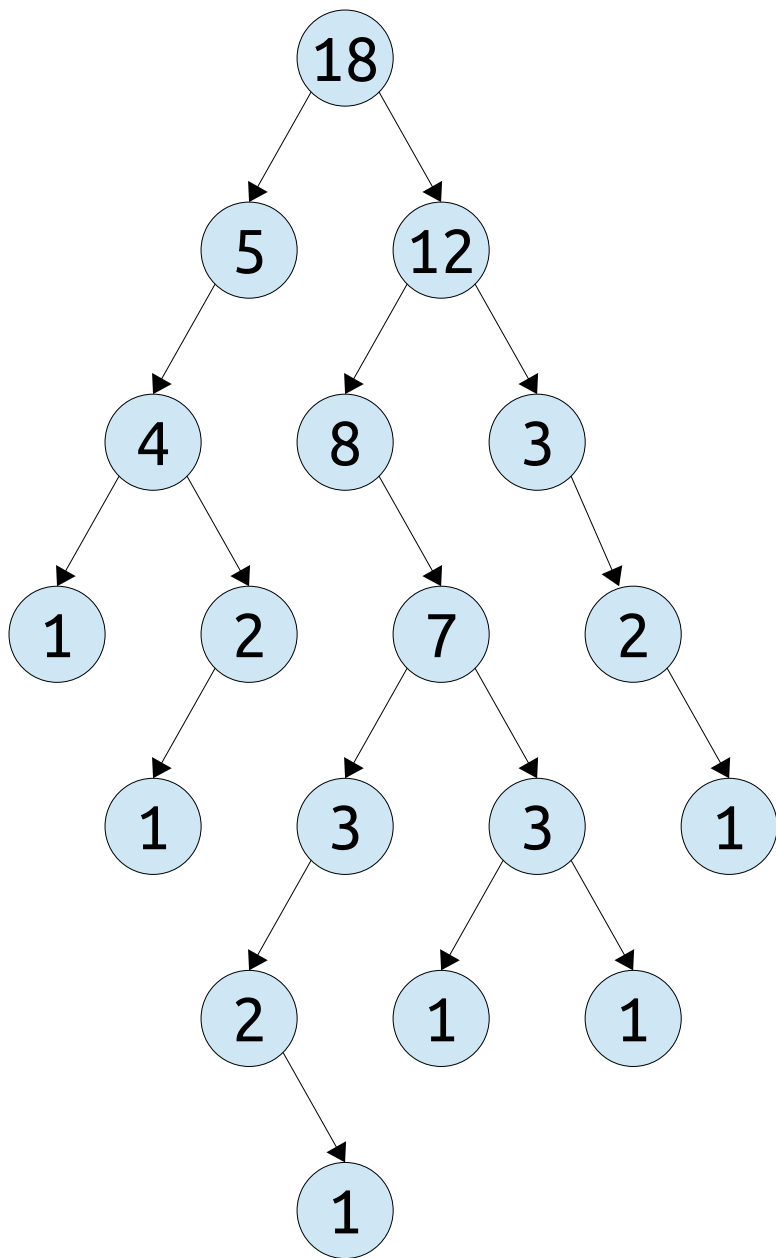
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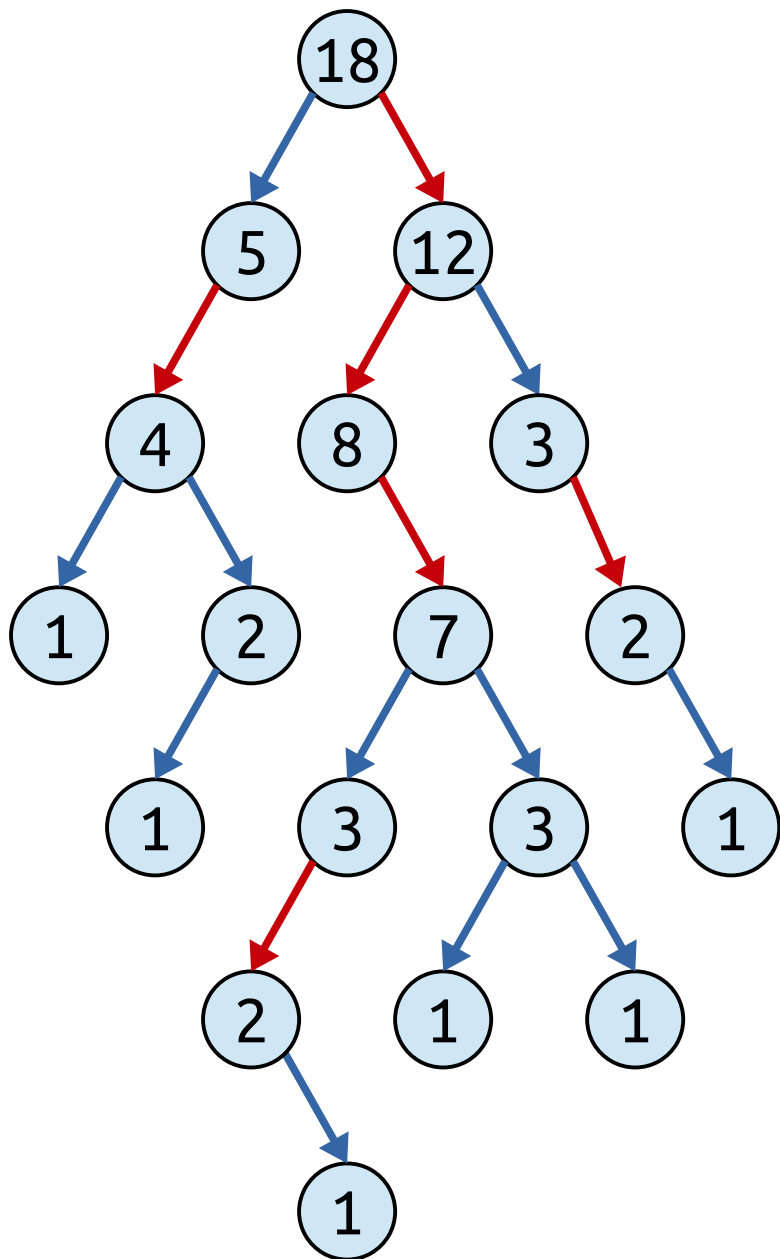
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Mark each edge as blue or red:

  $s(\text{child}) \leq \frac{1}{2} \cdot s(\text{parent})$   
  $s(\text{child}) > \frac{1}{2} \cdot s(\text{parent})$



Blue edges make lots of progress.

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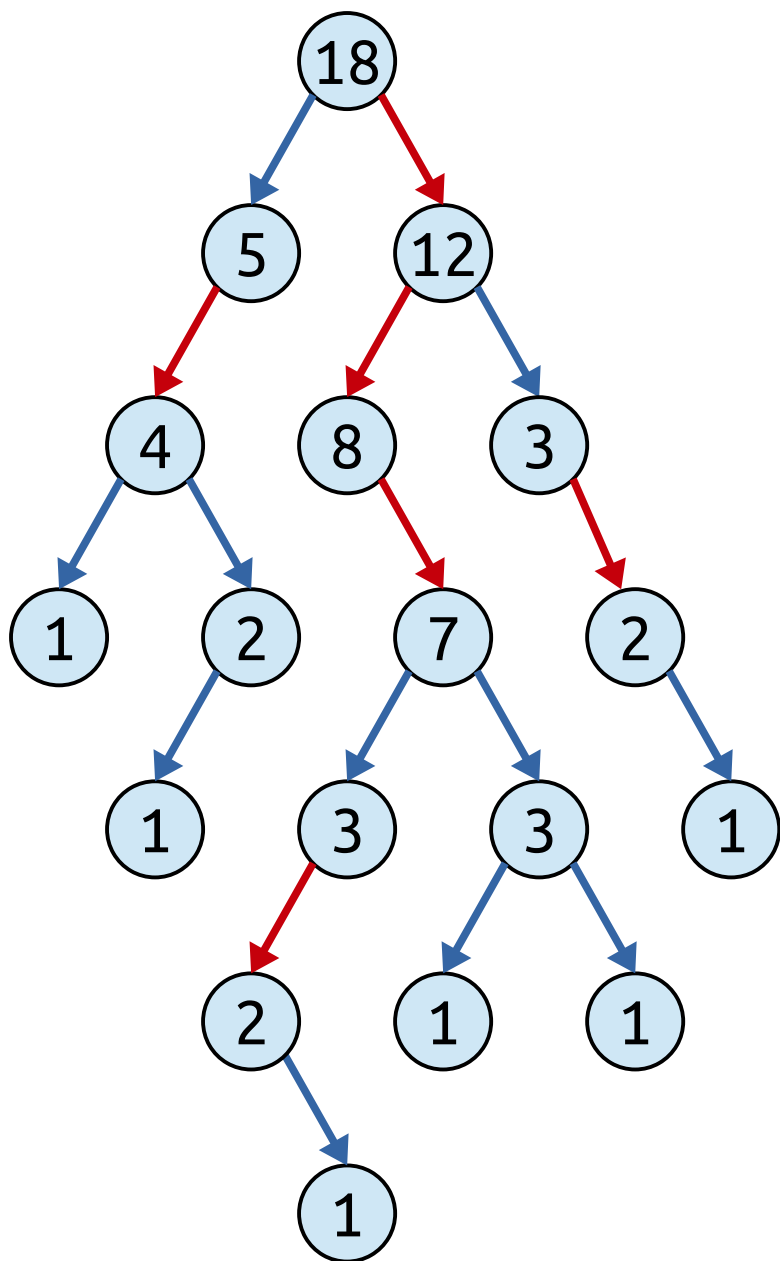
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→  $s(\text{child}) \leq \frac{1}{2} \cdot s(\text{parent})$

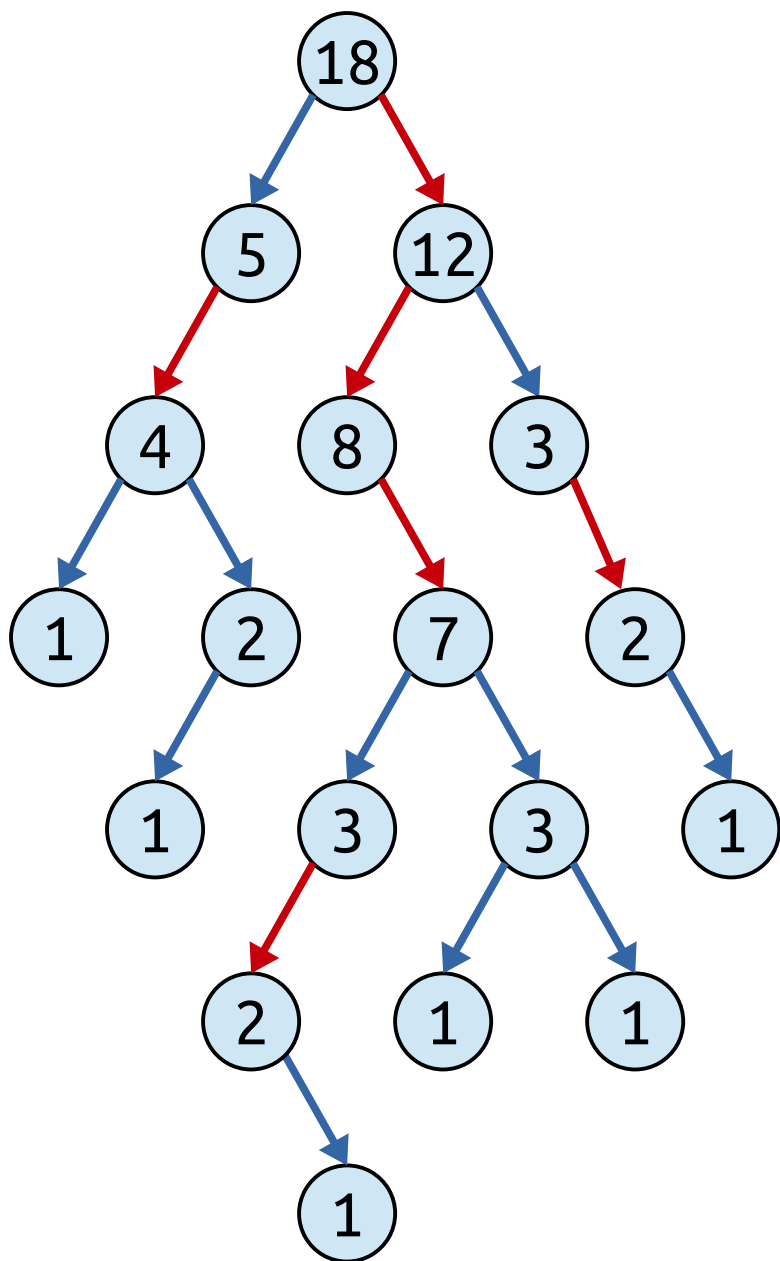
→  $s(\text{child}) > \frac{1}{2} \cdot s(\text{parent})$

Blue edges make lots of progress.

Cost of visiting a node:

$O(\text{\#blue-used} + \text{\#red-used})$

**Question:** Why is splaying fast?



Let  $s(x)$  denote the number of nodes in the subtree rooted at  $x$ .

Mark each edge as blue or red:

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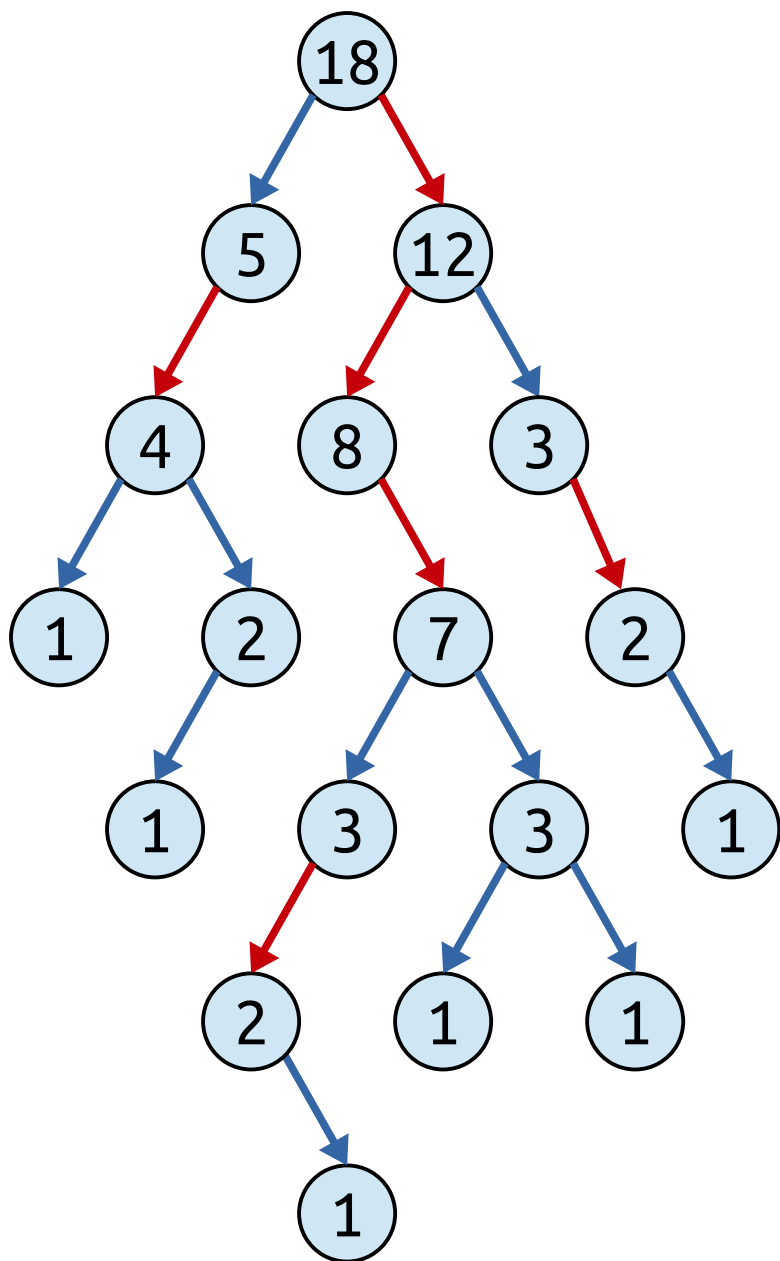
Cost of visiting a node:

$O(\text{\#blue-used} + \text{\#red-used})$

**Idea:** Bound the cost of blue edges, then amortize away the cost of red edges. This is called a **heavy/light decomposition**.



**Question:** Why is splaying fast?





Let  $s(x)$  denote the number of nodes in the subtree rooted at  $x$ .

Mark each edge as blue or red:

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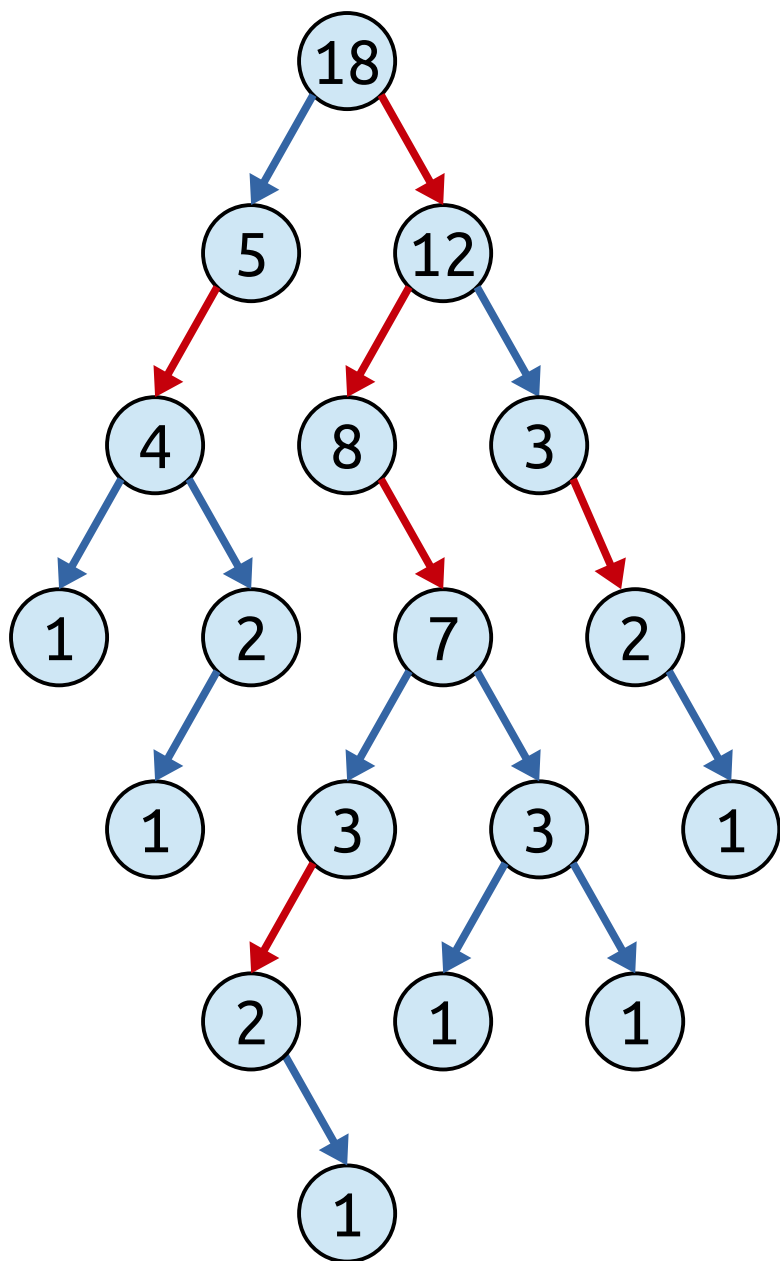
Cost of visiting a node:

$O(\text{\#blue-used} + \text{\#red-used})$

Give a bound on  $O(\text{\#blue-used})$  and explain your reasoning.

Answer at <https://pollev.com/cs166spr23>

**Question:** Why is splaying fast?



Let  $s(x)$  denote the number of nodes in the subtree rooted at  $x$ .

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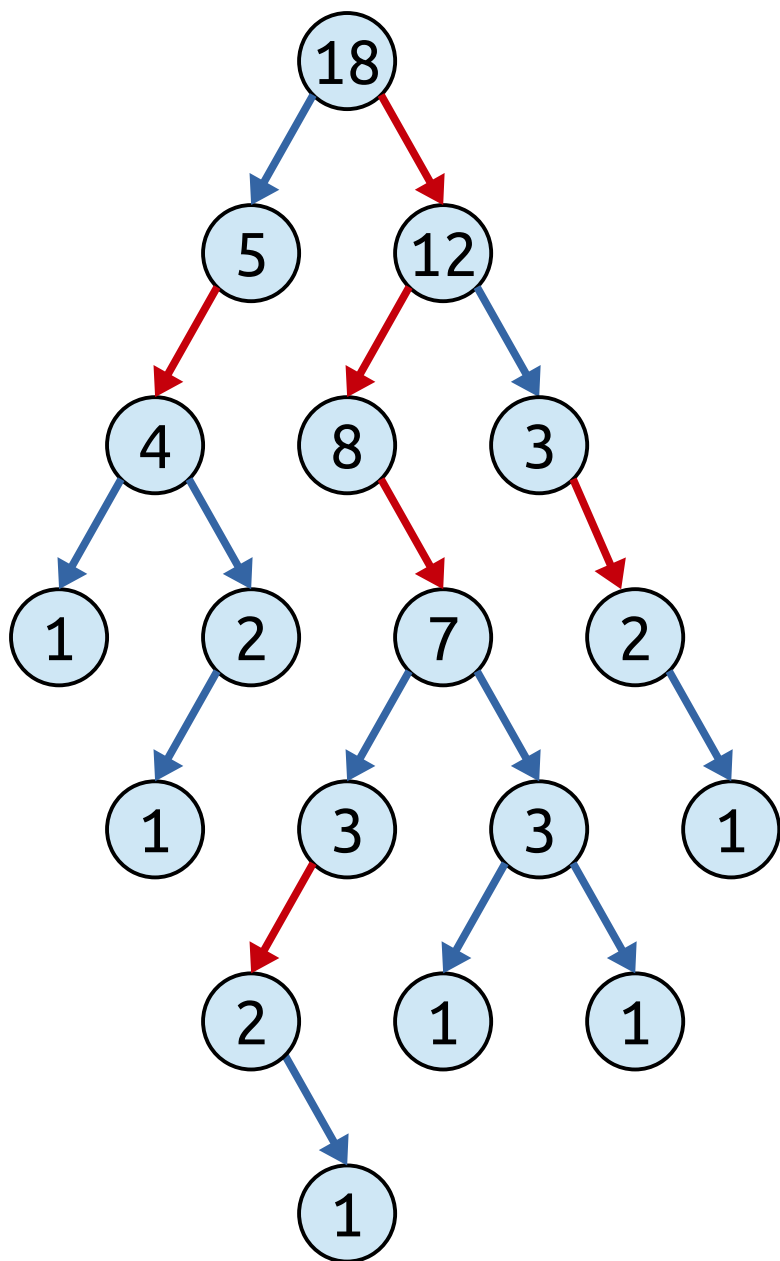
Blue edges make lots of progress.

Cost of visiting a node:

$O(\log n + \text{\#red-used})$

**Intuition:** Blue edges discard half the remaining nodes. You can only do that  $O(\log n)$  times before running out of nodes.

**Question:** Why is splaying fast?



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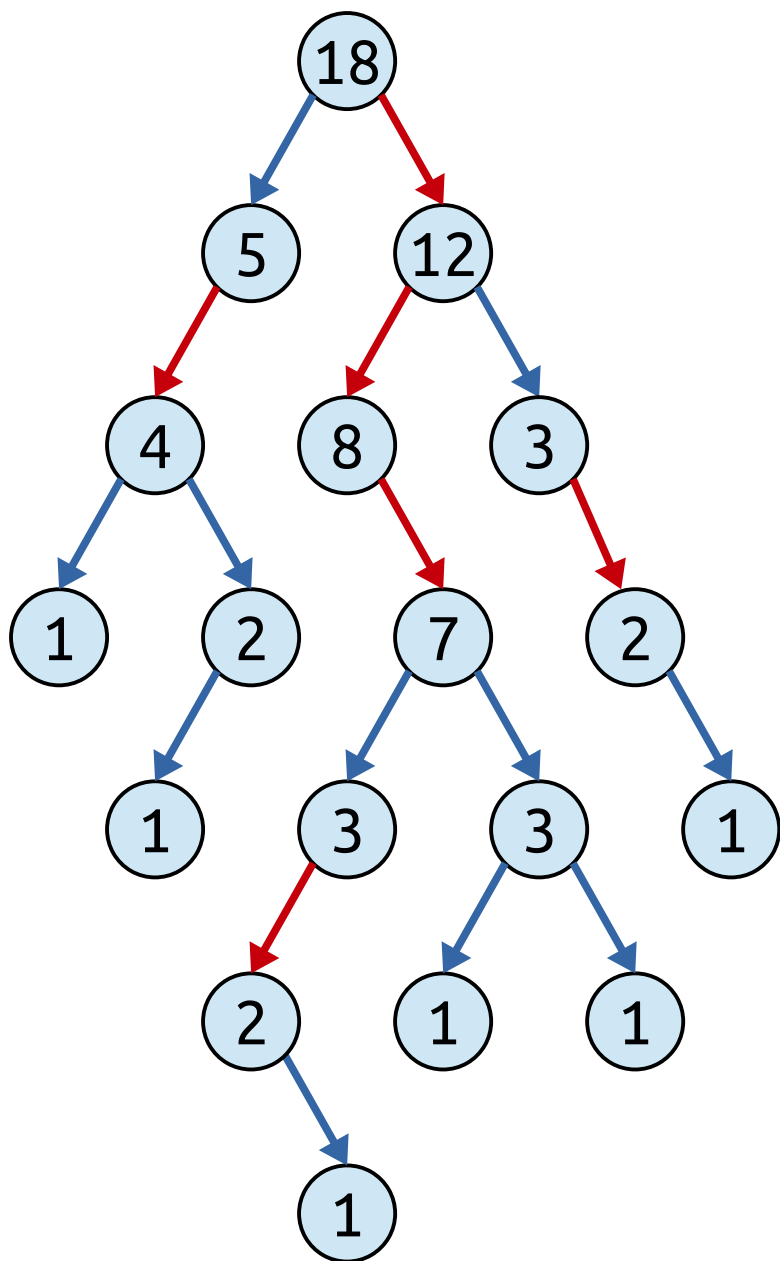
Blue edges make lots of progress.

Cost of visiting a node:

$$O(\log n + \text{\#red-used})$$

**Goal:** Find a potential function that penalizes red edges and rewards blue edges.

**Question:** Why is splaying fast?



Let  $s(x)$  denote the number of nodes in the subtree rooted at  $x$ .

Mark each edge as blue or red:

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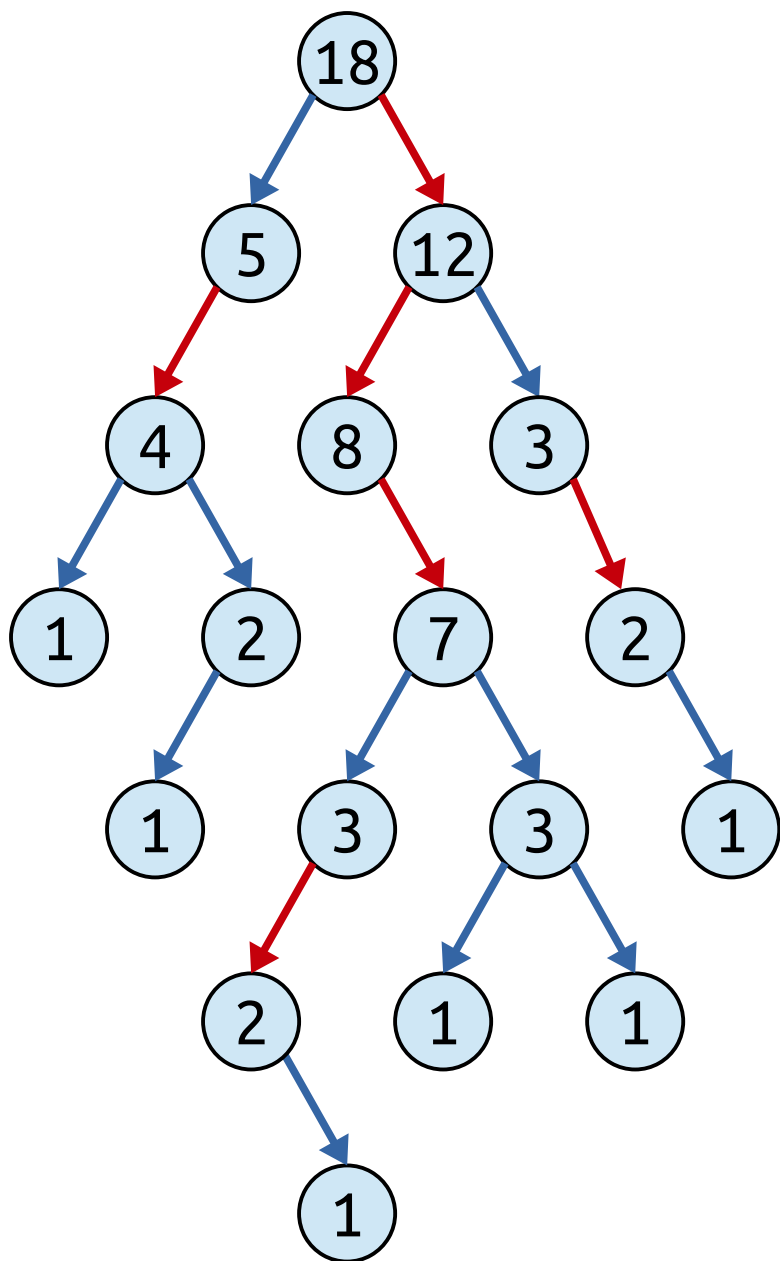
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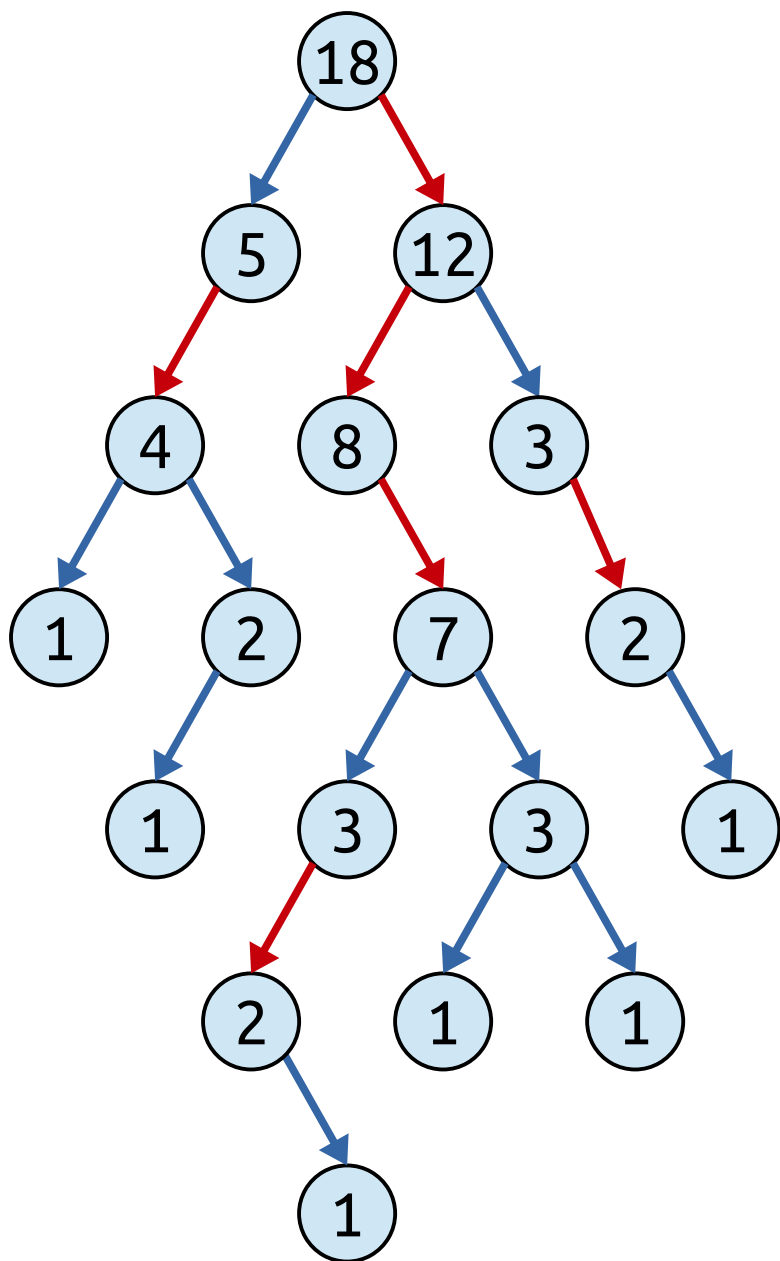
Blue edges make lots of progress.

Cost of visiting a node:

$O(\log n + \text{\#red-used})$

**Observation:** If there are a lot of red edges, then  $\lg s(x)$  will frequently be large.

**Question:** Why is splaying fast?



Let  $s(x)$  denote the number of nodes in the subtree rooted at  $x$ .

Mark each edge as blue or red:

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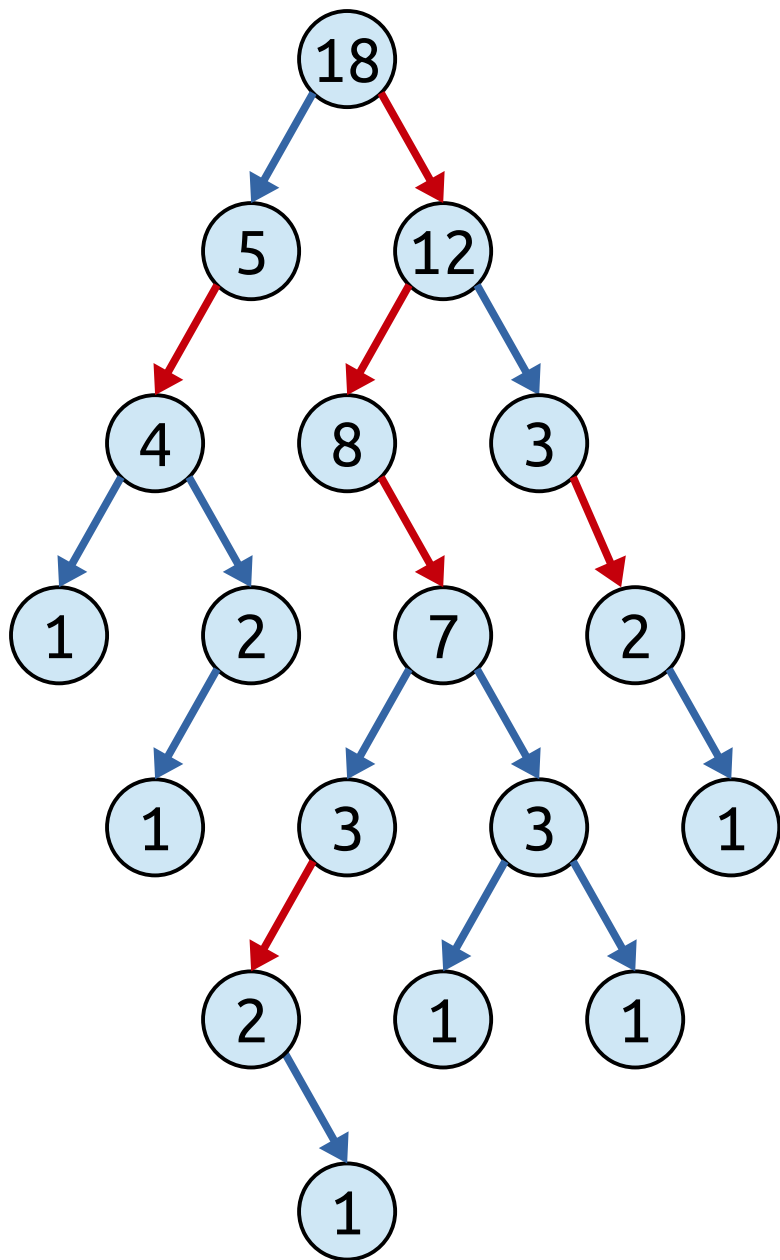
Cost of visiting a node:

$O(\log n + \text{\#red-used})$

Choose our potential to be

$$\Phi = \sum_{i=1}^n \lg s(x_i).$$

**Question:** Why is splaying fast?



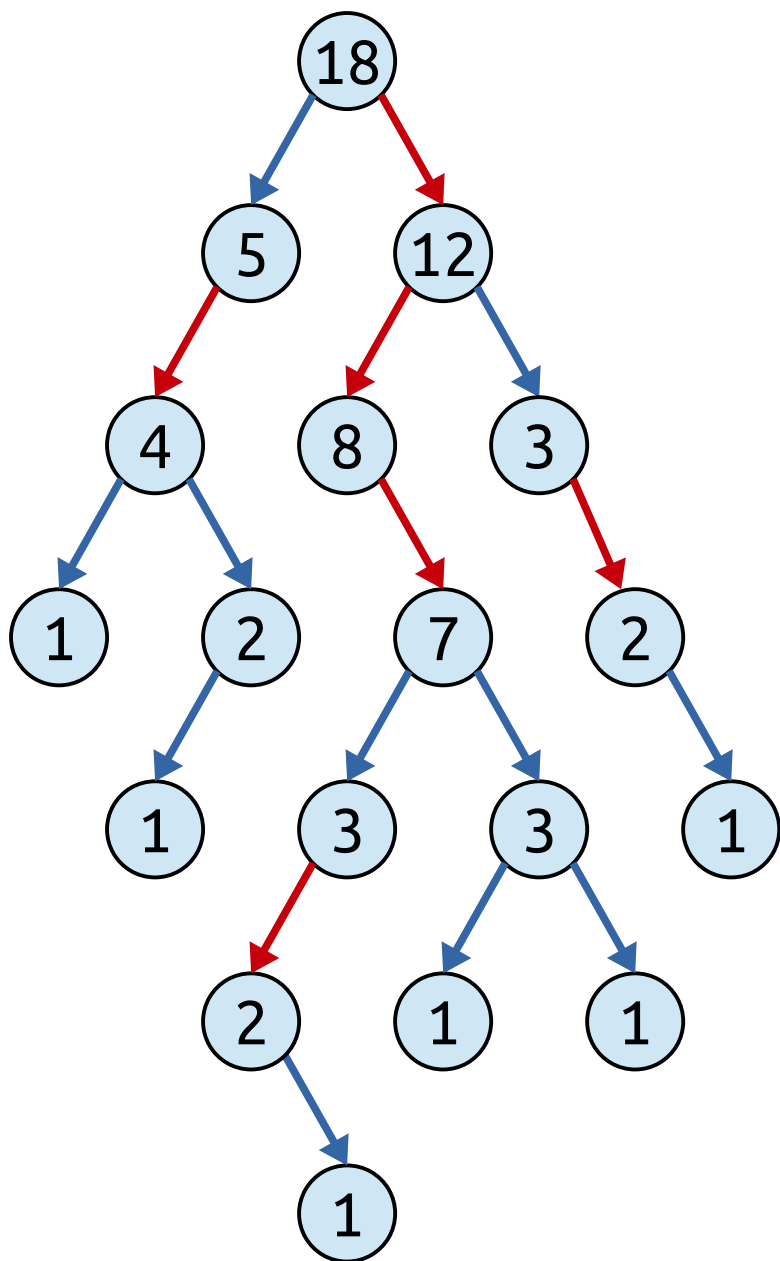
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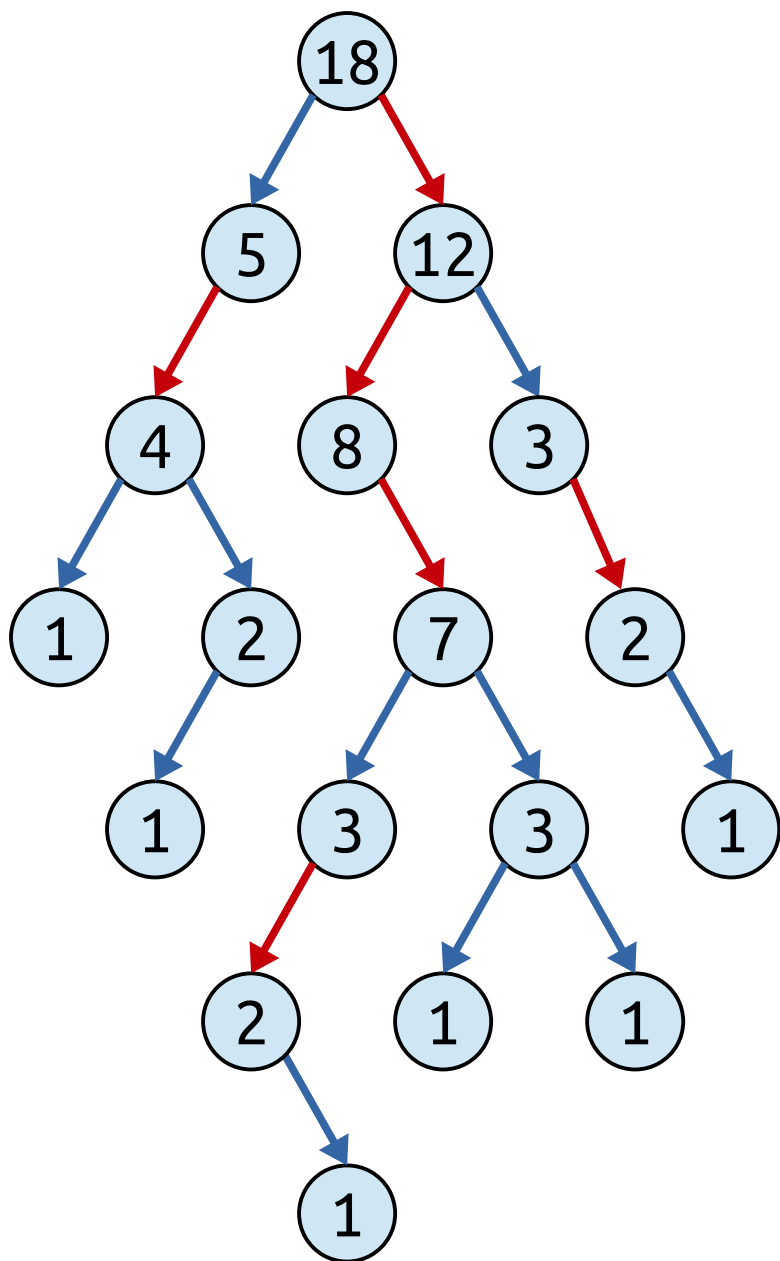
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Proving  $\Phi$  amortizes away the **\#red-used** term involves some detail-oriented math. Check the Sleator-Tarjan paper for details.

**Question:** Why is splaying fast?





Cost of visiting a node:

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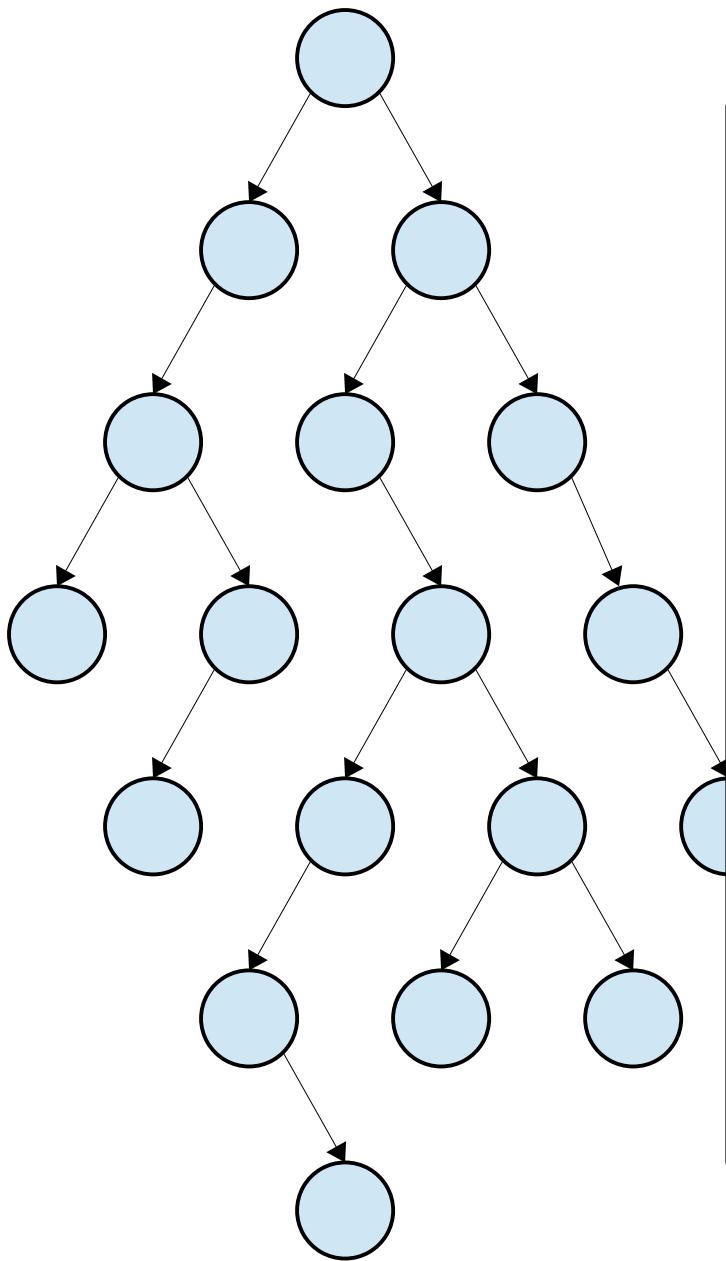
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**Theorem:** The amortized cost of a splay operation is  **$O(\log n)$** .

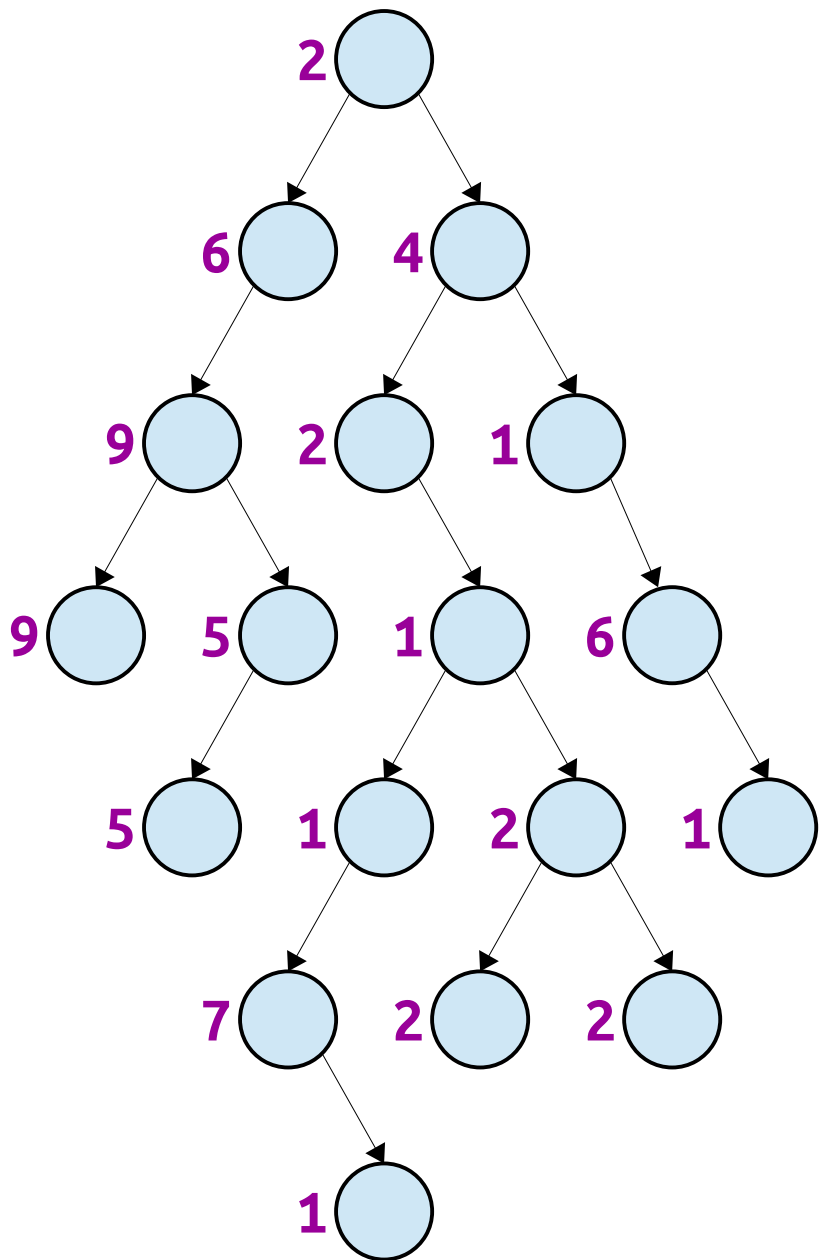
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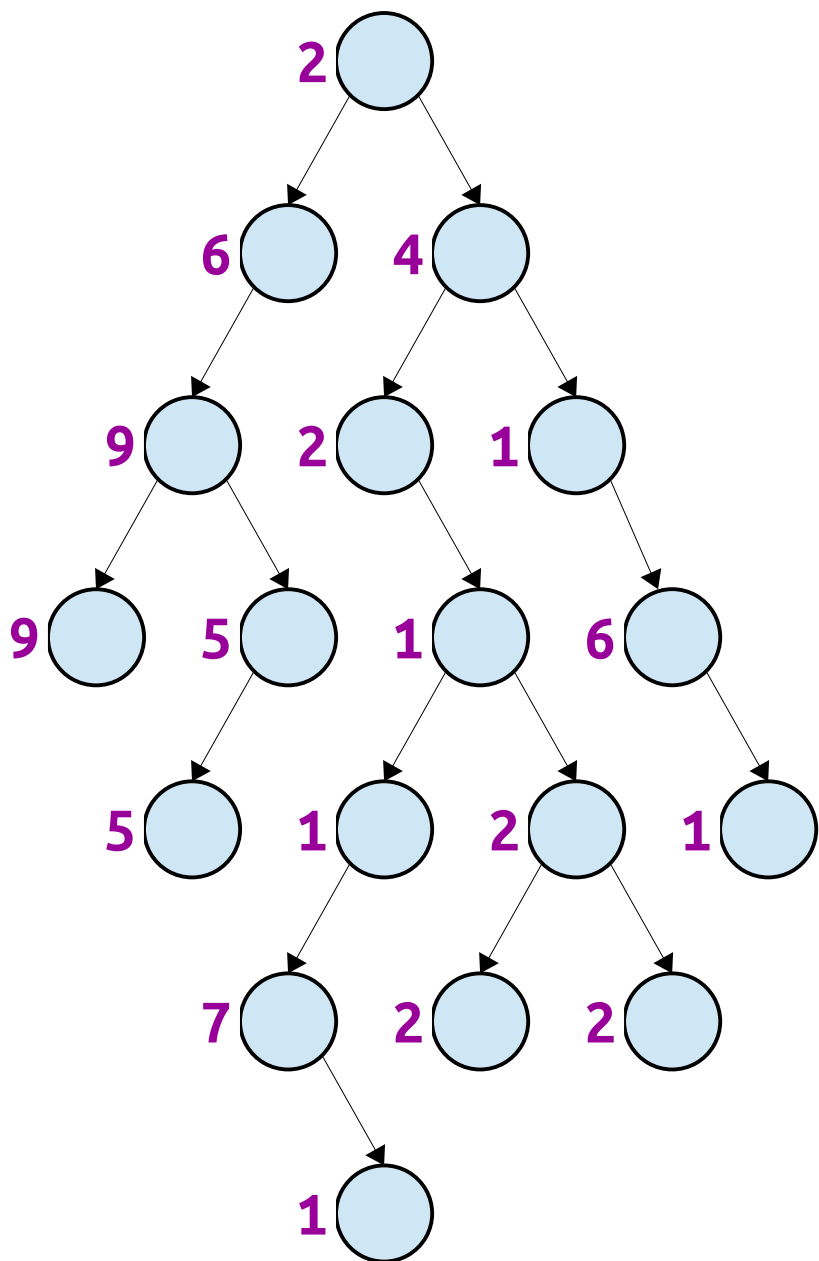
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**Question:** Why is splaying fast?

Some nodes are more important than others. Assign each a weight  $w_i$  and let the total weight be  $W$ .



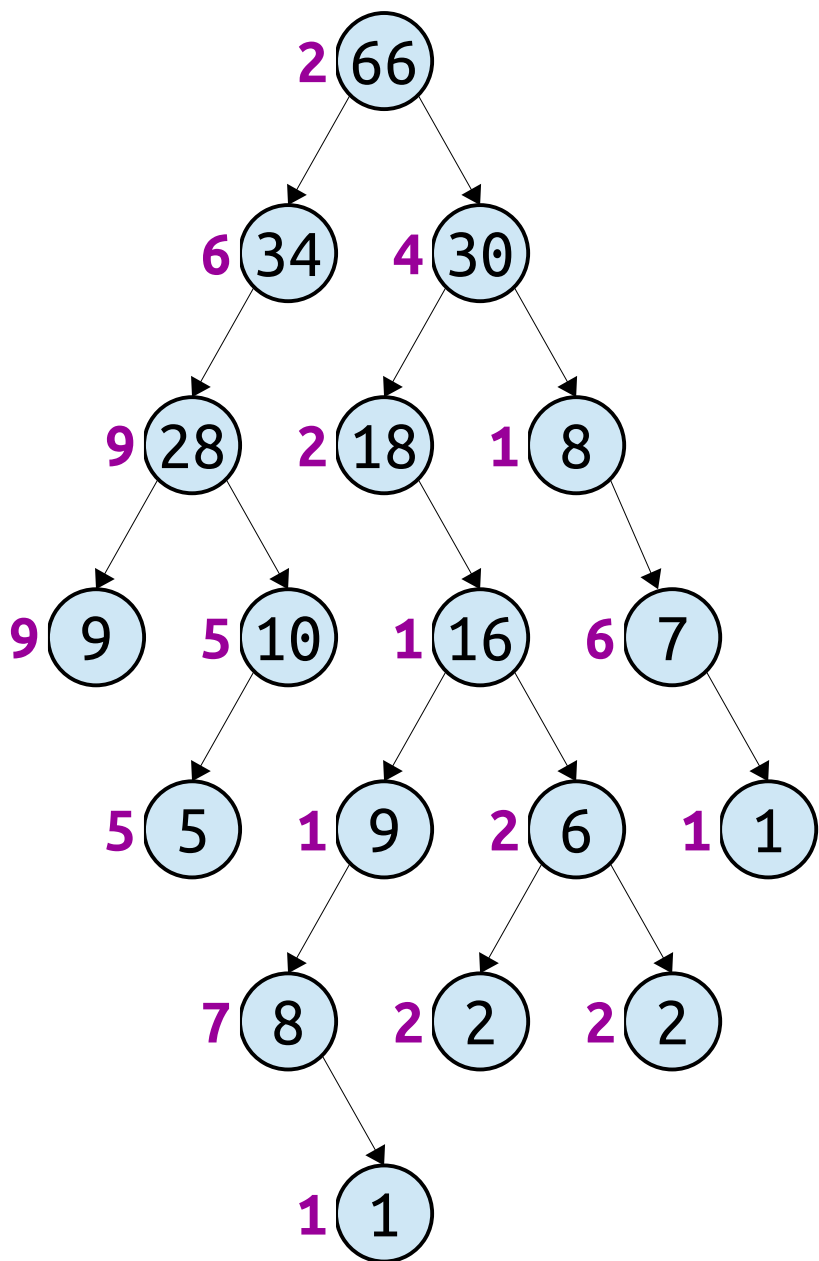
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Let  $s(x_i)$  be the sum of the weights in the tree rooted at  $x_i$ .

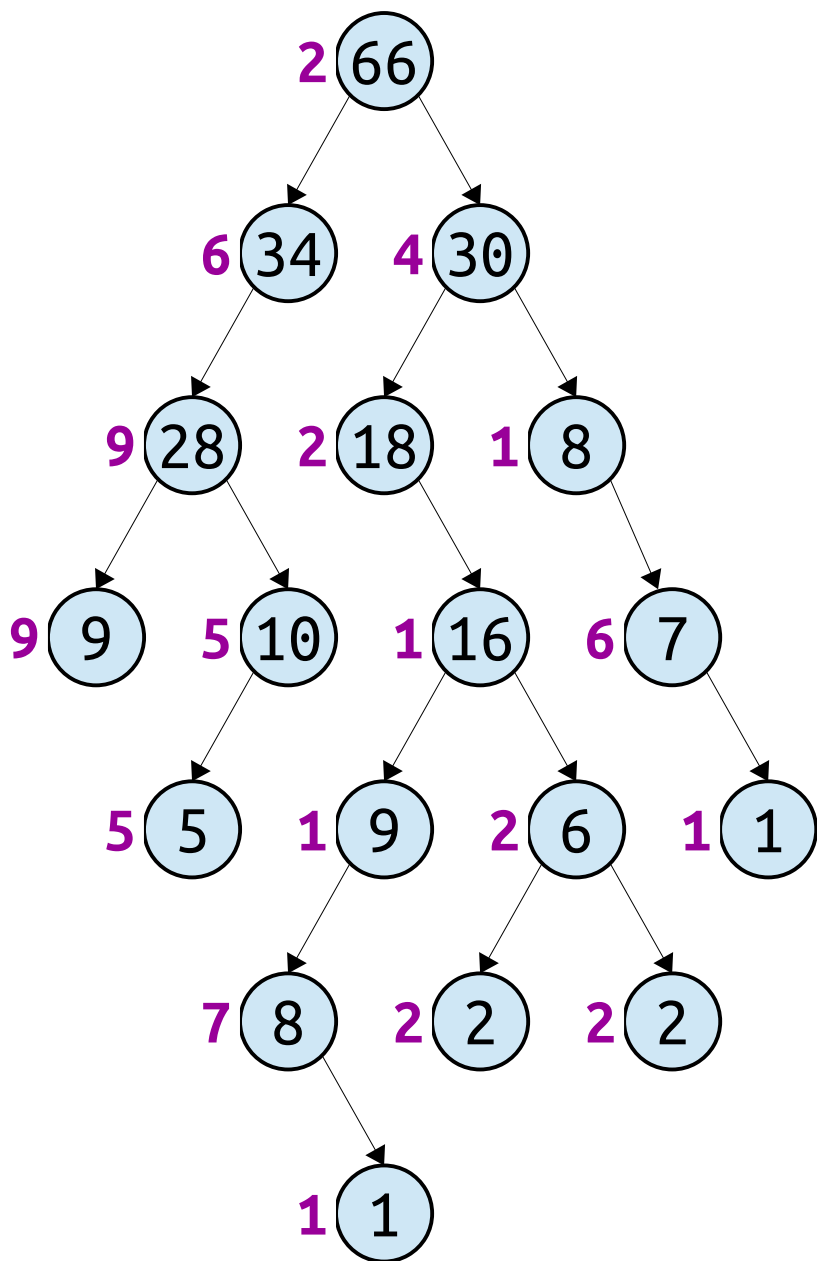
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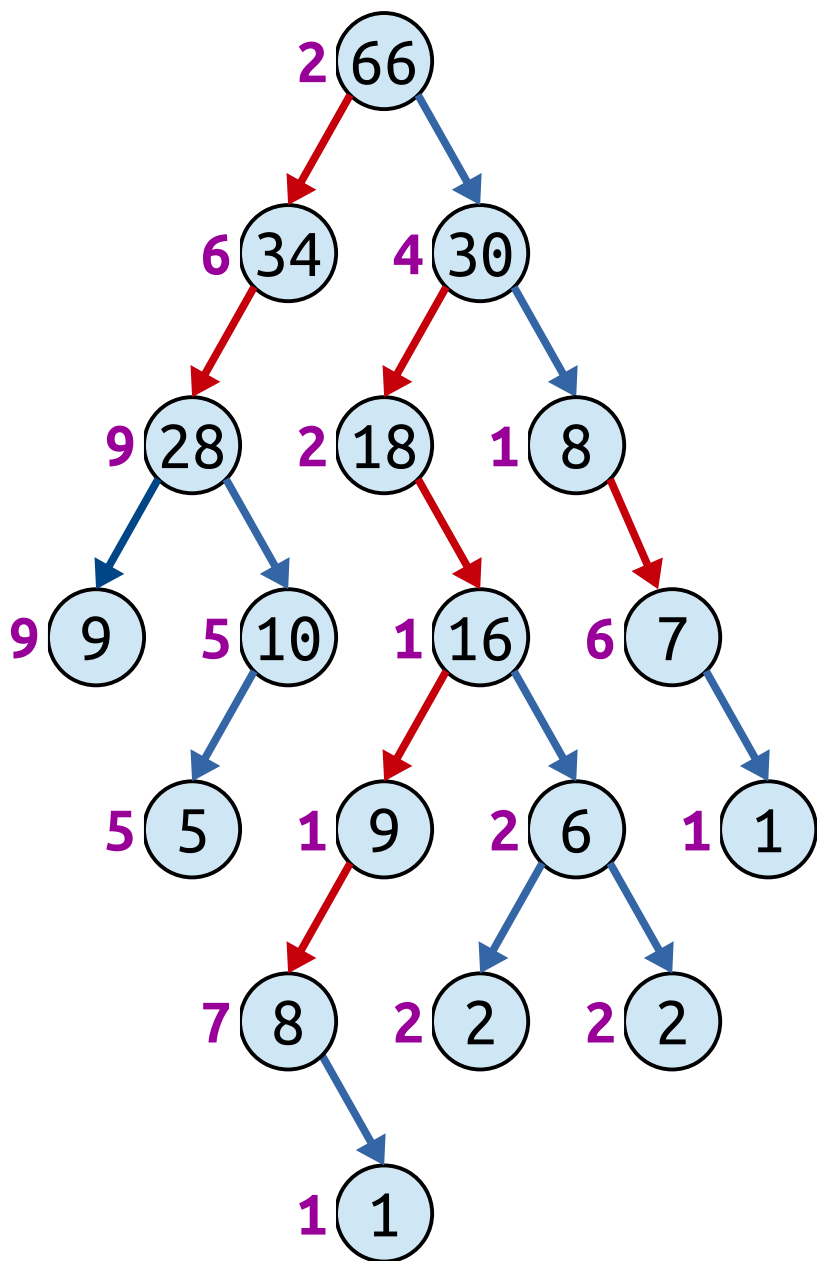
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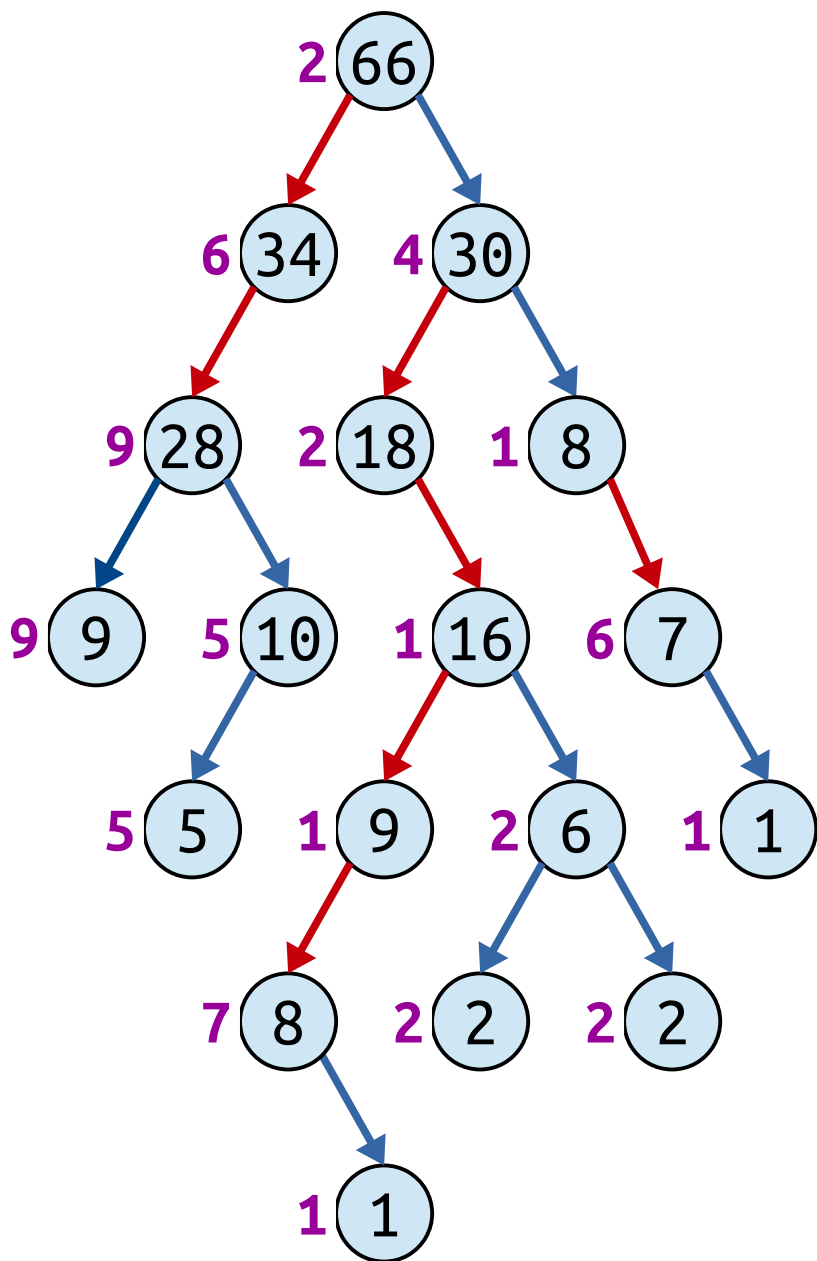
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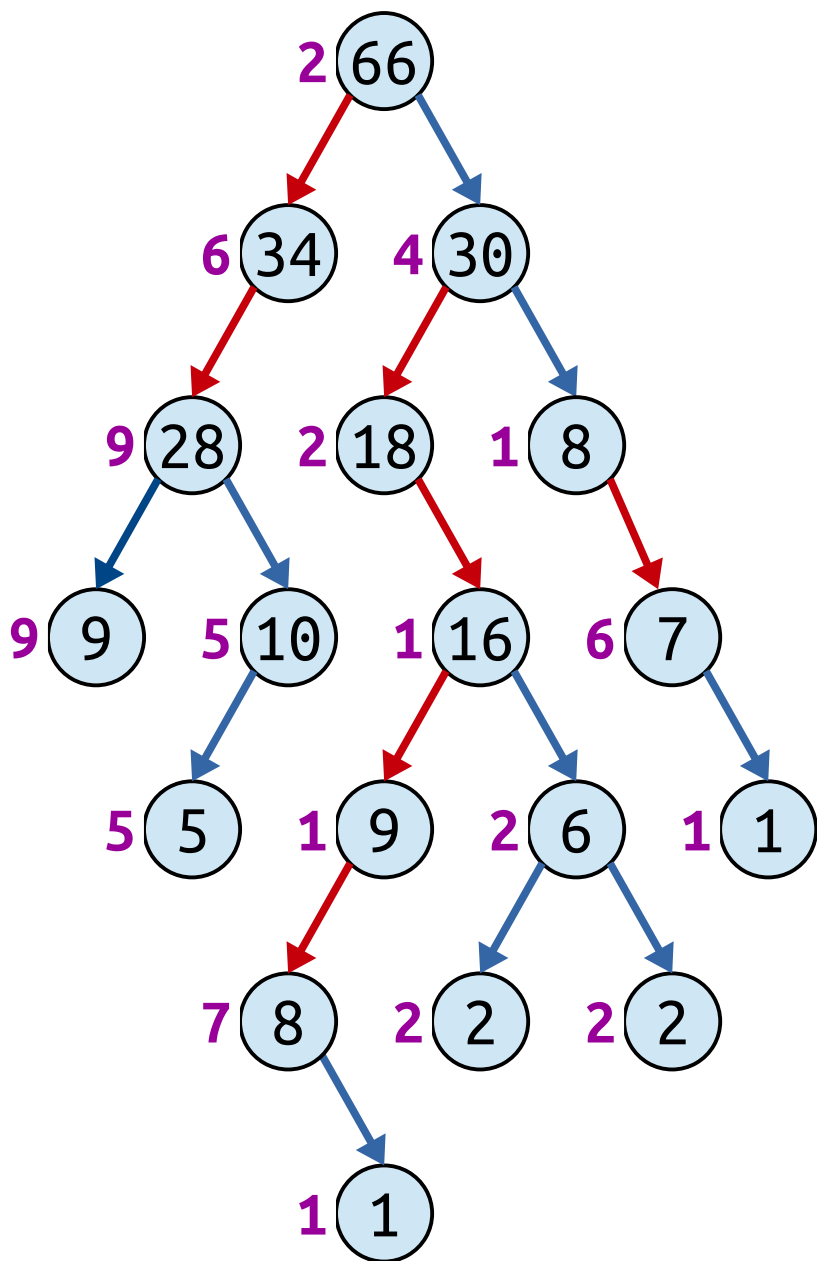
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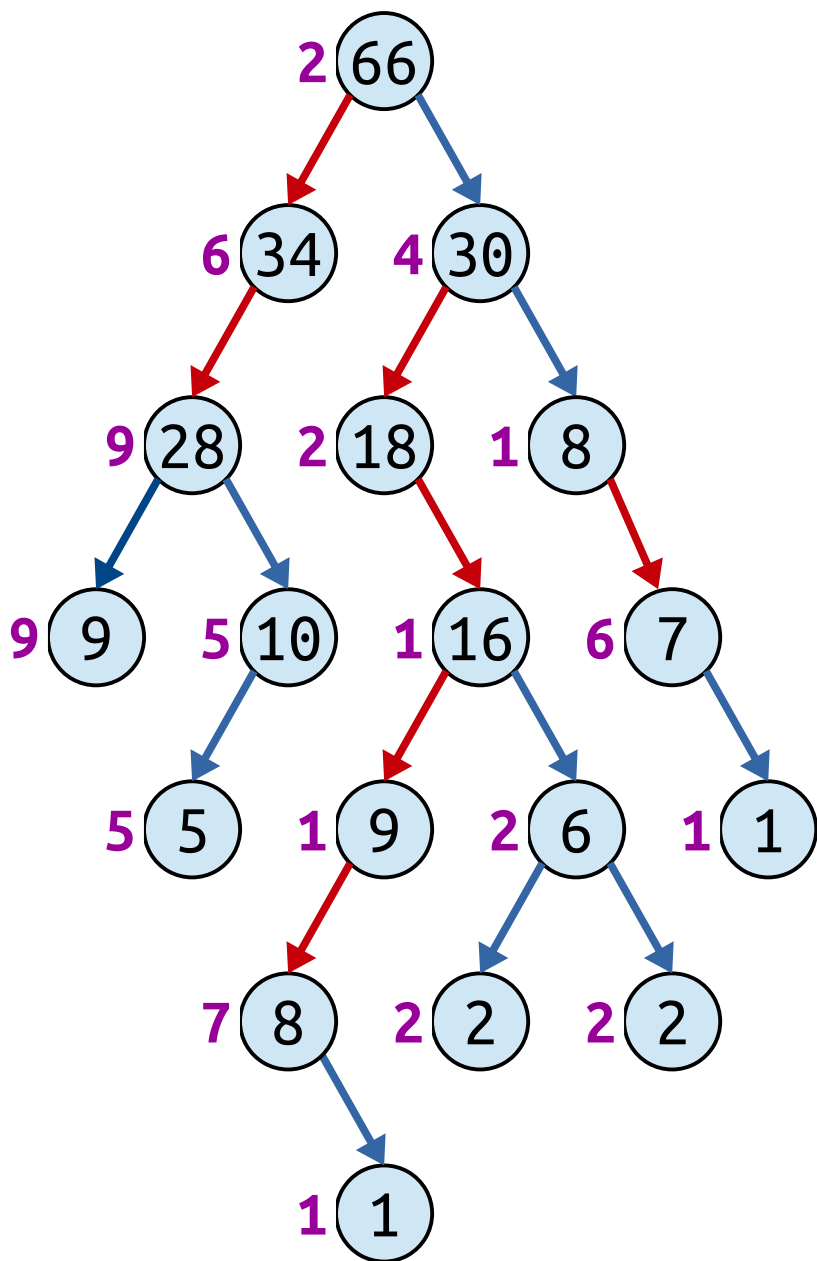
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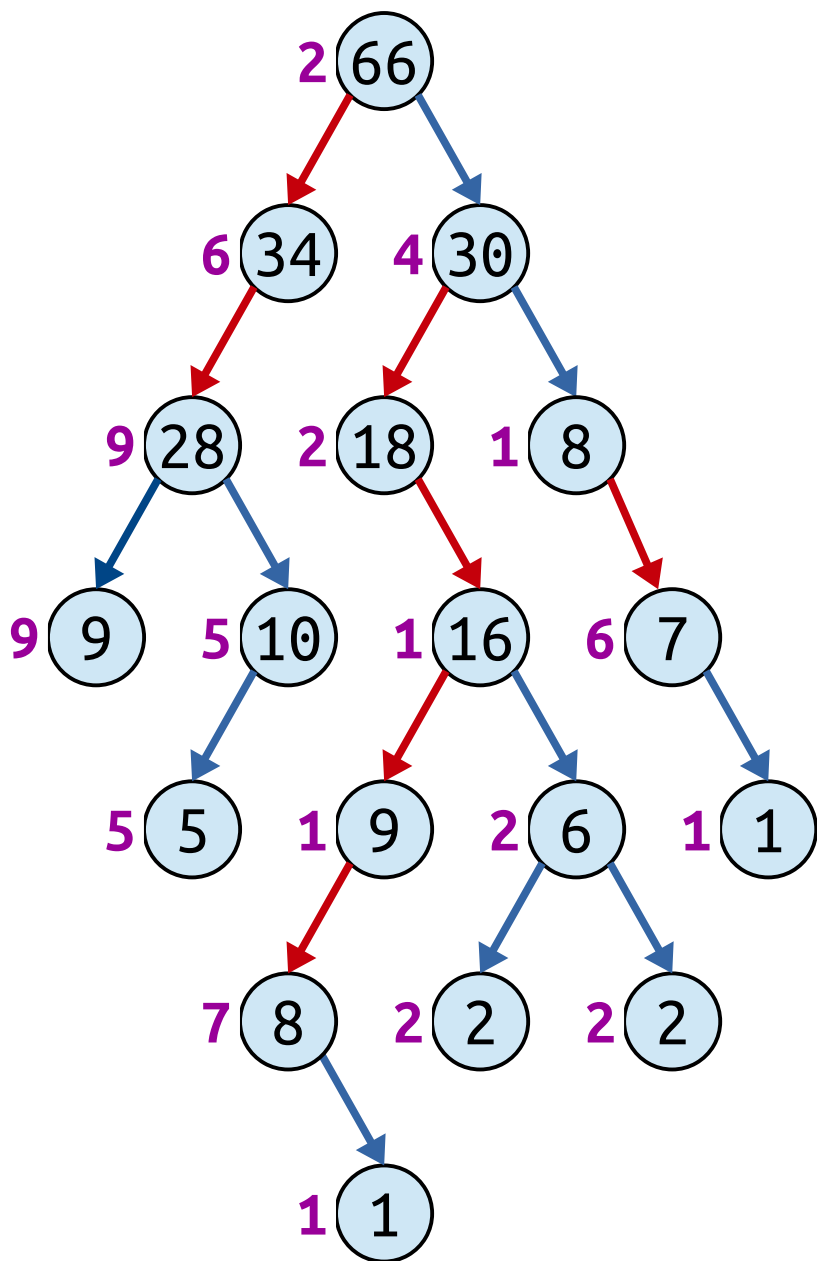
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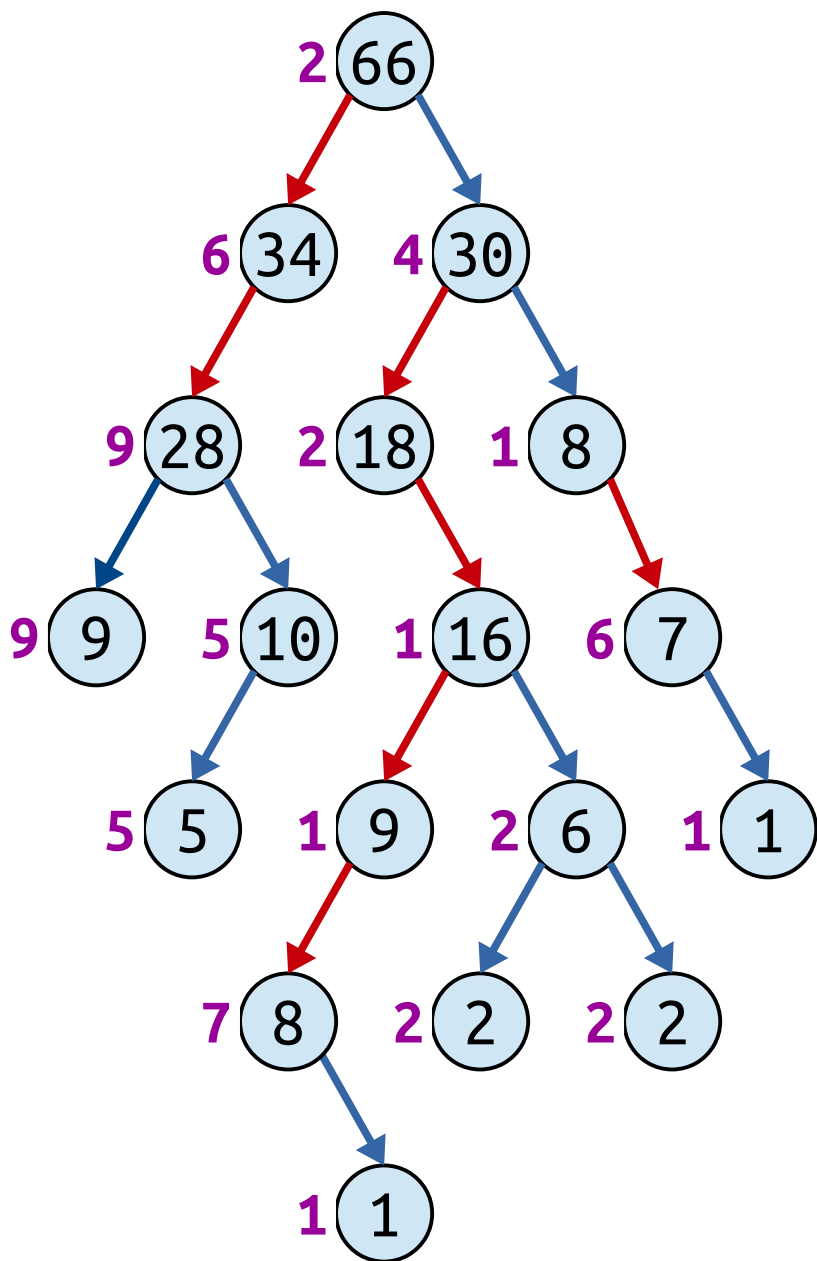
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Cost of visiting a node:

$O(\log(W / w_i) + \text{\#red-used})$

**Question:** Why is splaying fast?



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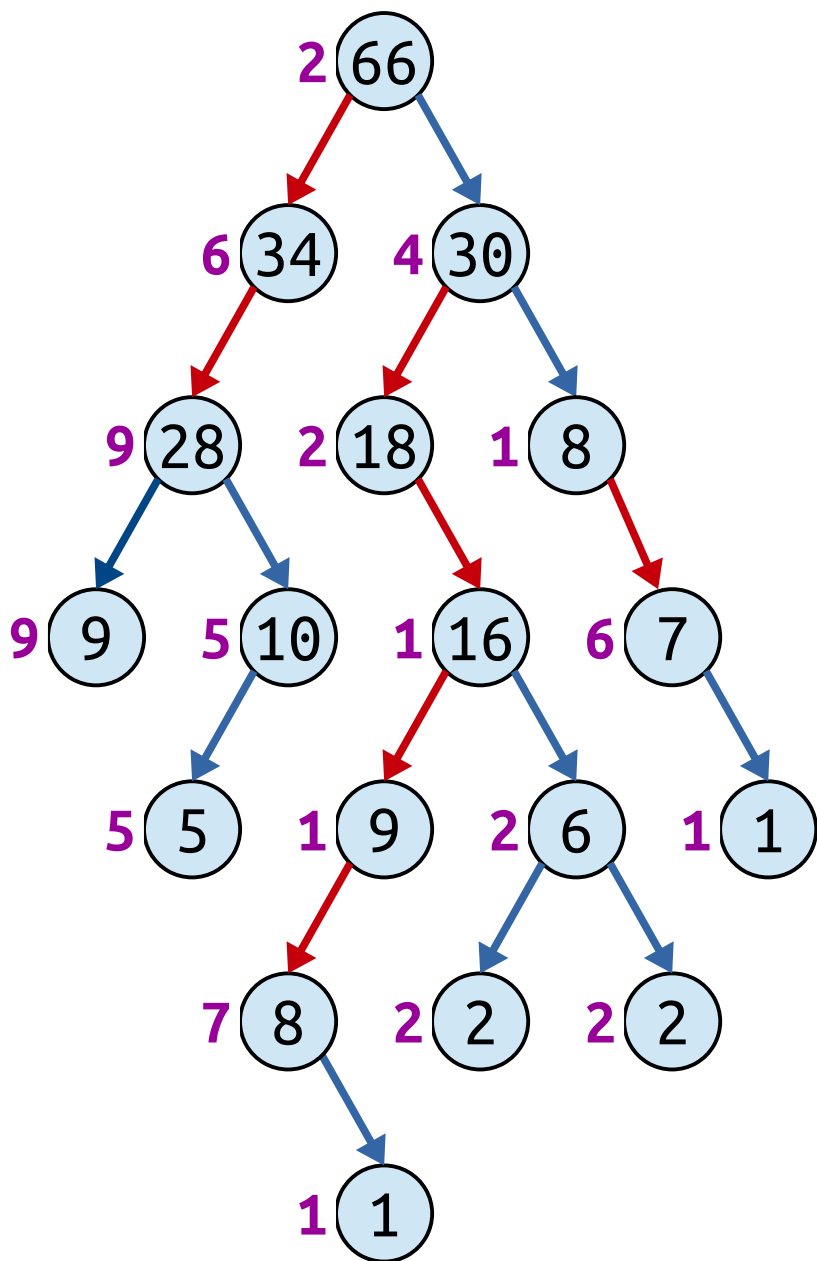
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Set  $\Phi = \sum_{i=1}^n \lg s(x_i)$ .

**Question:** Why is splaying fast?



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**Theorem:** The amortized cost of a splay is  $O(1 + \log(W / w_i))$ .

$$\text{Set } \Phi = \sum_{i=1}^n \lg s(x_i).$$

**Question:** Why is splaying fast?

# The Access Lemma

- Assign weights  $w_1, \dots, w_n$  to the nodes in the tree.
  - These weights are purely for accounting purposes and don't actually appear anywhere on the tree.
- Let  $W = w_1 + \dots + w_n$ .
- ***Lemma:*** The amortized cost of splaying at a node  $x_i$  is

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  - These weights are purely for accounting purposes and don't actually appear anywhere on the tree.
- Let  $W = w_1 + \dots + w_n$ .
- **Lemma:** The amortized cost of splaying at a node  $x_i$  is

$$O(1 + \log W + \log (1/w_i))$$

The total weight across all nodes should be small to keep this term small.

The weight of frequent items should be high to make this term low.



<i>Property</i>	<i>Description</i>
<i>Balance</i>	Lookups take time $O(\log n)$ .
<i>Entropy</i>	Lookups take expected time $O(1 + H)$ .
<i>Dynamic Finger</i>	Lookups take $O(\log \Delta)$ . $\Delta$ measures distance.
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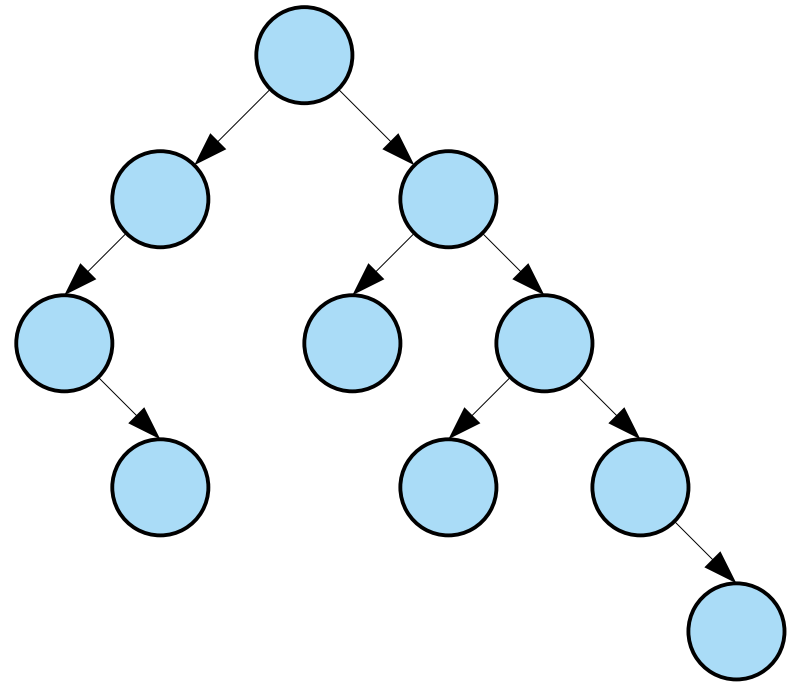
**Lemma:** Using the sum-of-logs potential, the amortized cost of splaying a node with weight  $w_i$  is  $O(1 + \log W + \log (1/w_i))$ , where  $W$  is the sum of all the weights.

**Balance Property:** The cost of any lookup in the binary search tree is  $O(\log n)$ , where  $n$  is the number of nodes.

Assign each node weight  $1/n$ .

$$W = 1$$
$$w_i = 1/n$$

Amortized cost of a lookup:  
 $O(1 + \log W + \log (1/w_i))$   
 $= O(1 + \log 1 + \log n)$   
 $= \mathbf{O(\log n)}$ .



**Lemma:** Using the sum-of-logs potential, the amortized cost of splaying a node with weight  $w_i$  is  $O(1 + \log W + \log (1/w_i))$ , where  $W$  is the sum of all the weights.



**Entropy Property:** Expected cost of a lookup is  $O(1 + H)$ , assuming lookups are drawn from a fixed distribution.

$$H = \sum_{i=1}^n p_i \lg \frac{1}{p_i}.$$

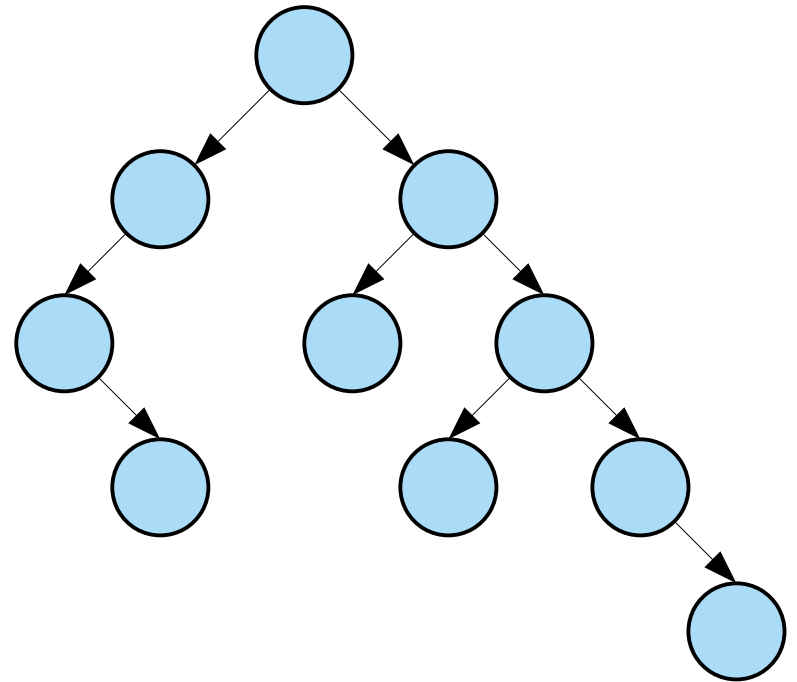
Pick  $w_i = p_i$ .

$$W = 1.$$

Cost of looking up key  $x_i$ :

$$\begin{aligned} & O(1 + \log W + \log (1/w_i)) \\ &= O(1 + \log 1 + \log (1/p_i)) \\ &= \mathbf{O(1 + \log (1/p_i))} \end{aligned}$$

So expected cost is  $O(1 + H)$ .



**Lemma:** Using the sum-of-logs potential, the amortized cost of splaying a node with weight  $w_i$  is  $O(1 + \log W + \log (1/w_i))$ , where  $W$  is the sum of all the weights.

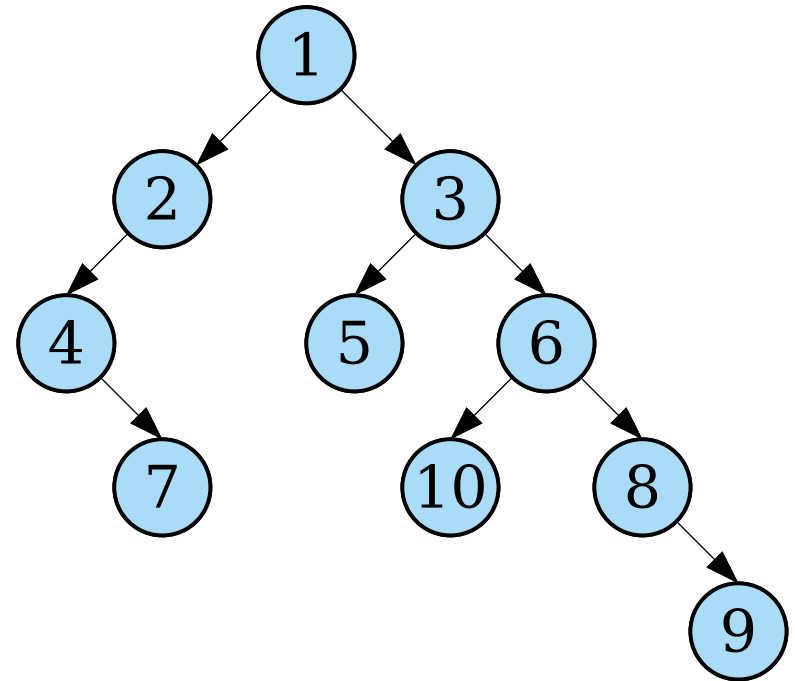
### **Working Set Property:**

Lookups take time  $O(\log t)$ , where  $t$  is the number of keys queried since the last time element was queried.

It doesn't immediately seem like we can use the theorem below, since the value of  $t$  depends on what accesses have been done recently.

For now, let's set that aside and focus on one snapshot in time.

Each key  $x_i$  is annotated with its value of  $t_i$ . How do we pick weights?



**Lemma:** Using the sum-of-logs potential, the amortized cost of splaying a node with weight  $w_i$  is  $O(1 + \log W + \log (1/w_i))$ , where  $W$  is the sum of all the weights.

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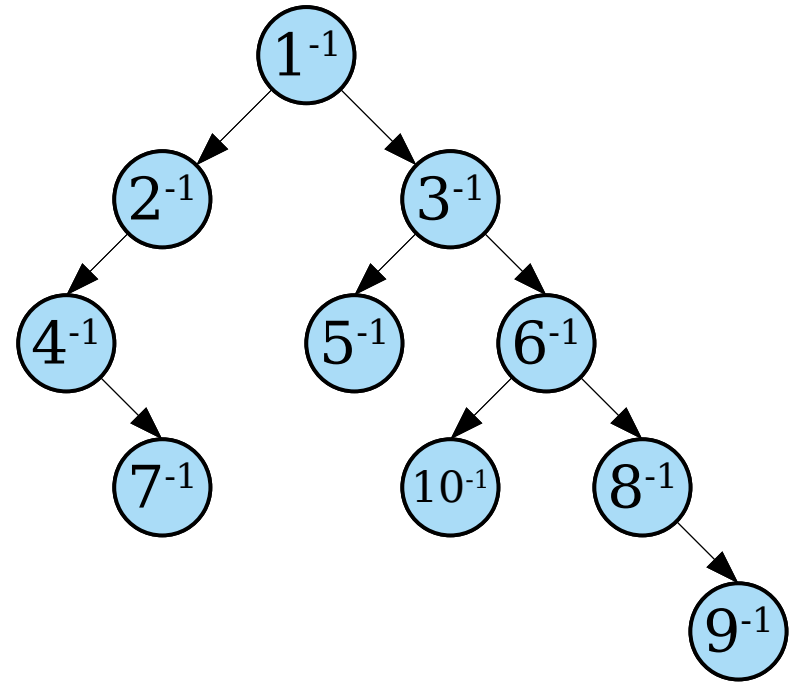
Reasoning by analogy:

**Balance:** Target is  $O(\log n)$ .  
Picked  $w_i = 1/n$ .

**Entropy:** Target is  $O(\log (1/p_i))$ .  
Picked  $w_i = p_i$ .

**Working Set:** Target is  $O(\log t_i)$ .  
Pick  $w_i = 1 / t_i$ .

**Question:** Does this work?



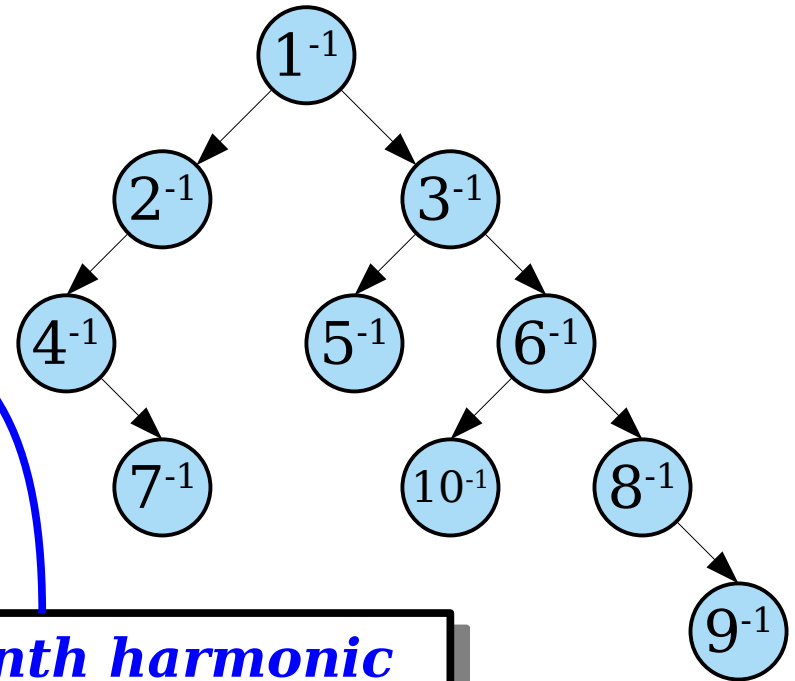
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### **Working Set Property:**

Lookups take time  $O(\log t)$ , where  $t$  is the number of keys queried since the last time element was queried.

$$w_i = 1 / t_i$$

$$W = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$
$$= \Theta(\log n)$$



This is the ***nth harmonic number***, denoted ***H<sub>n</sub>***.

Useful fact:  
 **$\ln(n+1) \leq H_n \leq (\ln n) + 1$** .

**Lemma:** Using the sum-of-logs potential, the amortized cost of splaying a node with weight  $w_i$  is  $O(1 + \log W + \log(1/w_i))$ , where  $W$  is the sum of all the weights.

### **Working Set Property:**

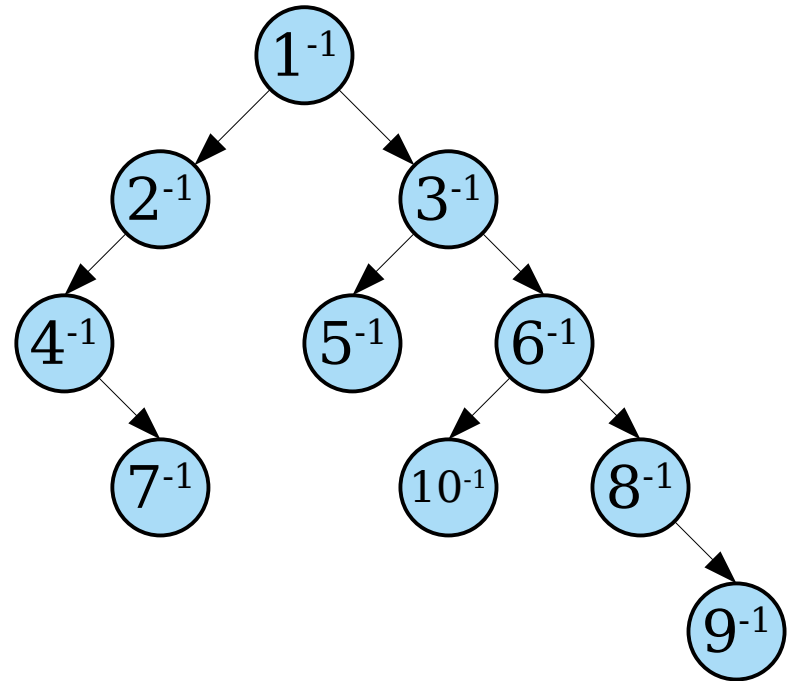
Lookups take time  $O(\log t)$ , where  $t$  is the number of keys queried since the last time element was queried.

$$w_i = 1 / t_i$$

$$W = 1/1 + 1/2 + 1/3 + \dots + 1/n \\ = \Theta(\log n)$$

$$O(1 + \log W + \log (1/w_i)) \\ = \mathbf{O(\log t_i + \log \log n)}.$$

Close! can we do better?



**Lemma:** Using the sum-of-logs potential, the amortized cost of splaying a node with weight  $w_i$  is  $O(1 + \log W + \log (1/w_i))$ , where  $W$  is the sum of all the weights.



### **Working Set Property:**

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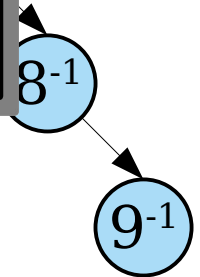
$$O(1 + \log W + \log (1/w_i))$$

$= \mathbf{O(\log t_i + \log \log n)}$ .

Close! can we do better?

The sum of the weights is too large for  $W$  to work out the way we want.

Can we pick weights so that  $W = O(1)$  and  $\log (1 / w_i) = O(\log t_i)$ ?



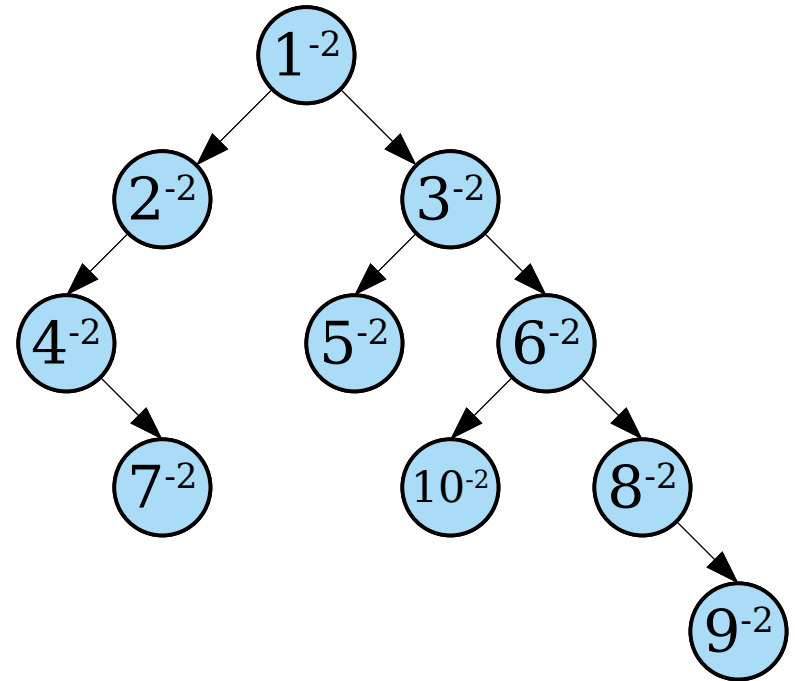
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### **Working Set Property:**

Lookups take time  $O(\log t)$ , where  $t$  is the number of keys queried since the last time element was queried.

$$w_i = 1 / t_i^2$$

$$W = 1/1^2 + 1/2^2 + 1/3^2 + \dots + 1/n^2$$



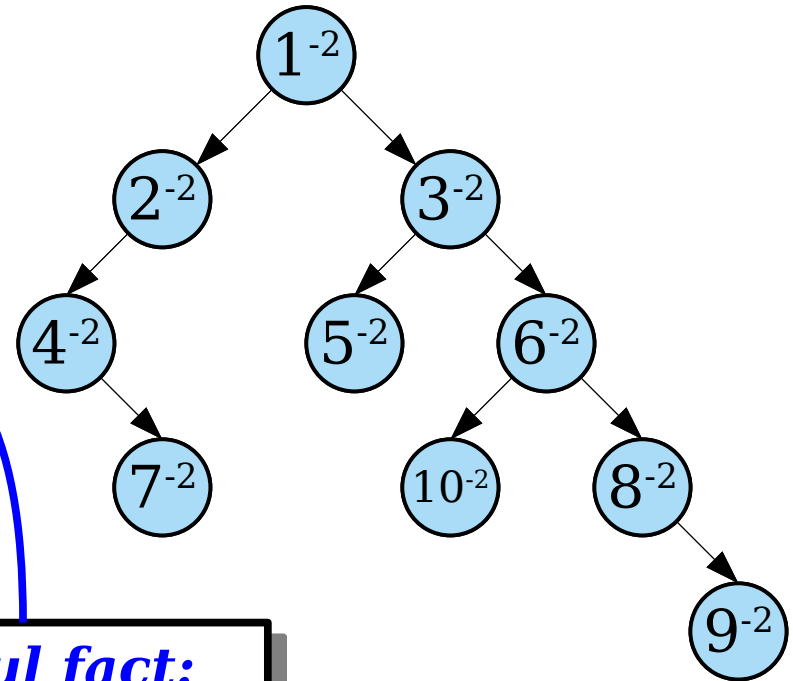
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Lookups take time  $O(\log t)$ , where  $t$  is the number of keys queried since the last time element was queried.

$$w_i = 1 / t_i^2$$

$$W = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} \\ = O(1)$$



### **Useful fact:**

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$$

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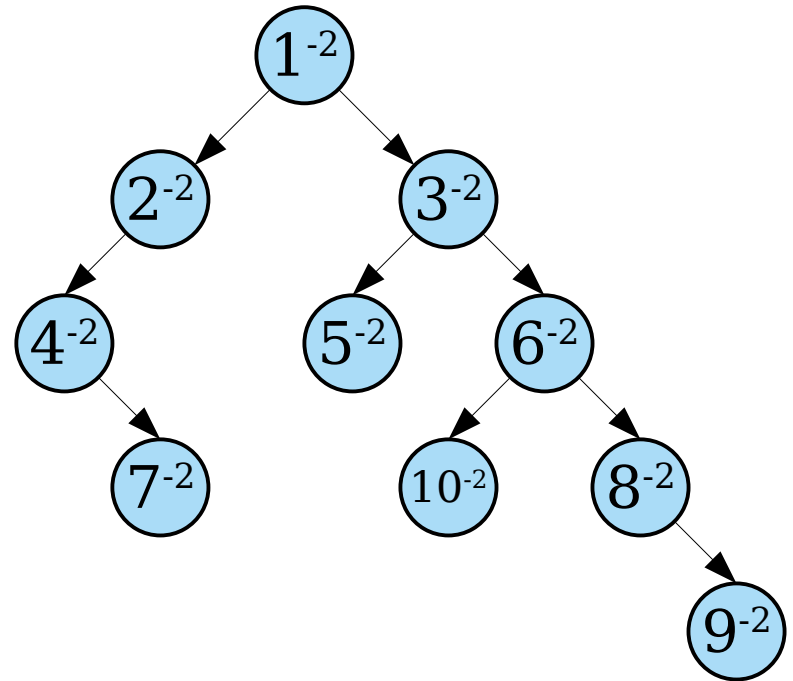
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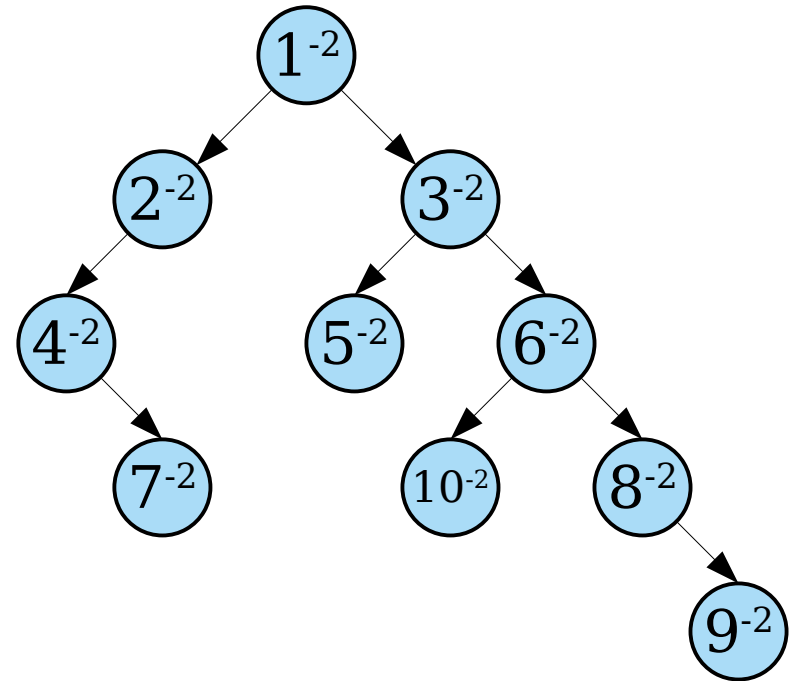
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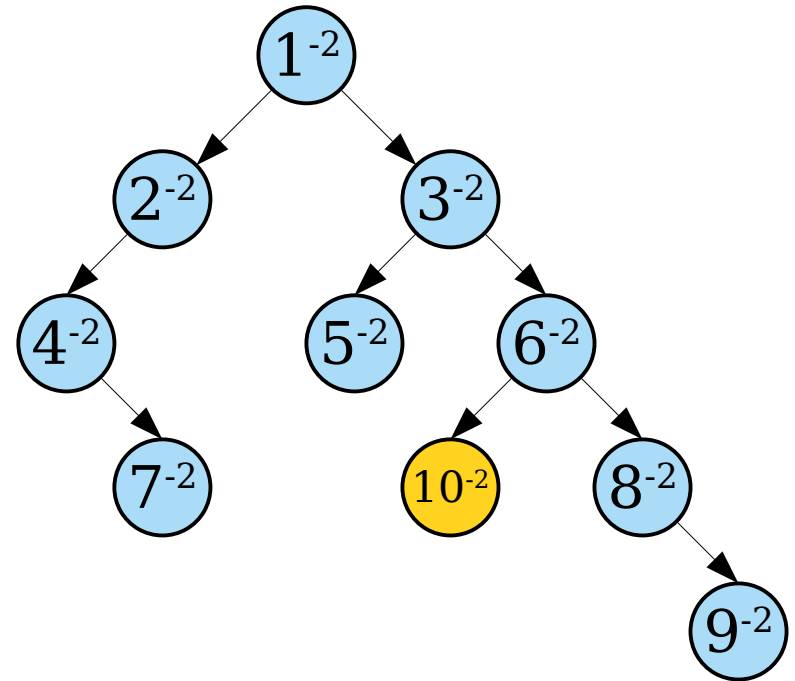
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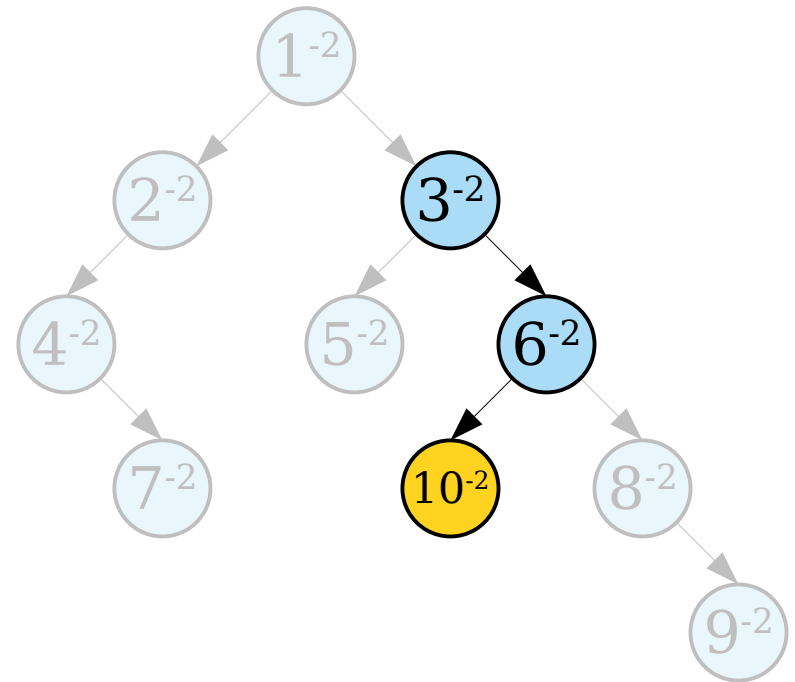
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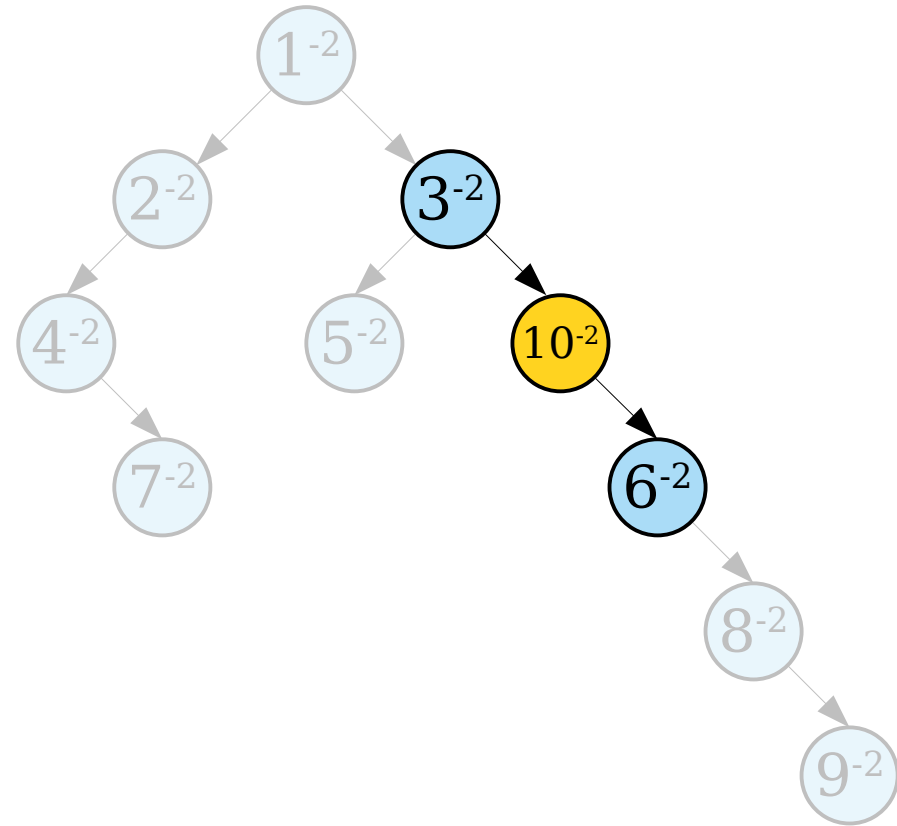
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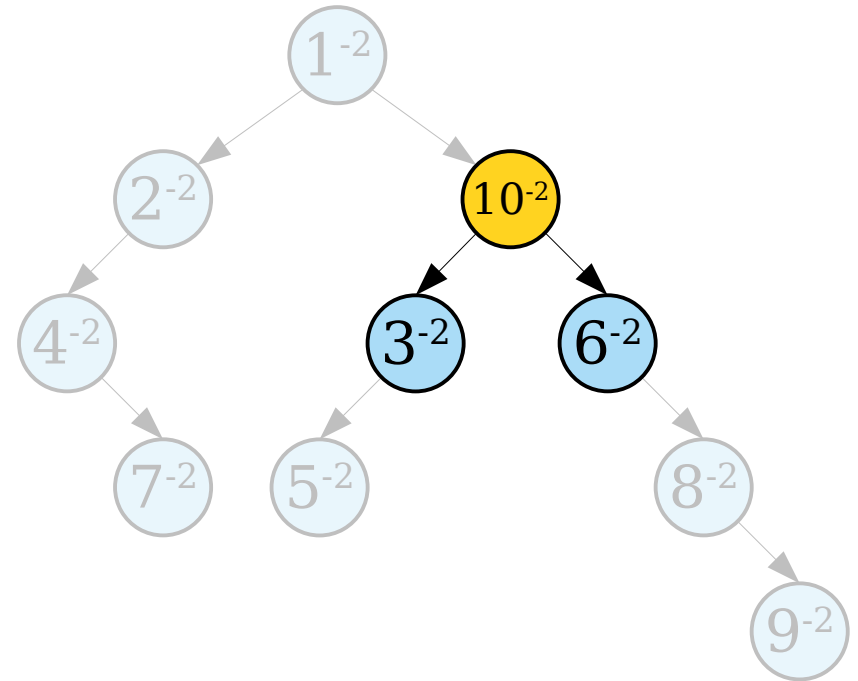


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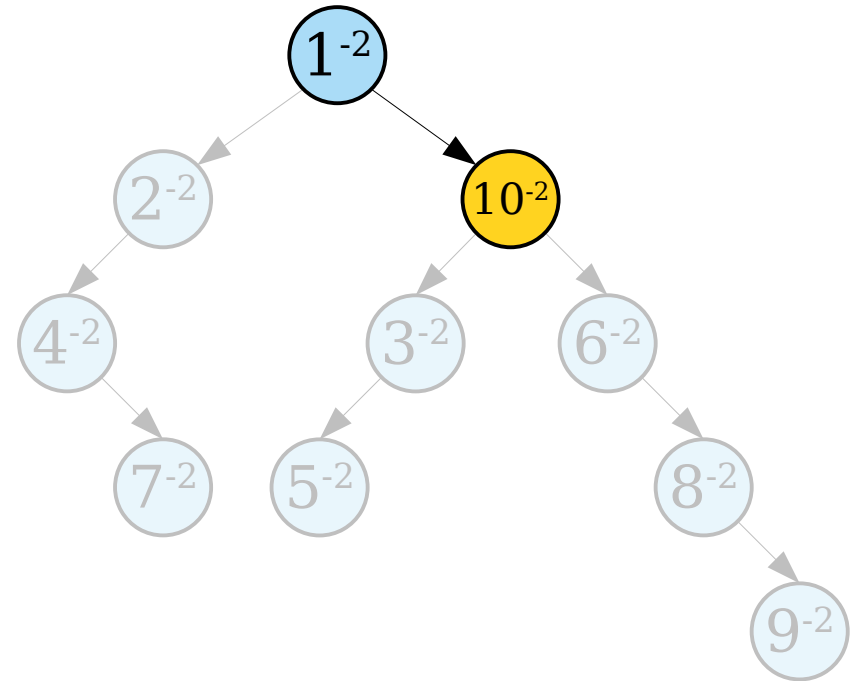
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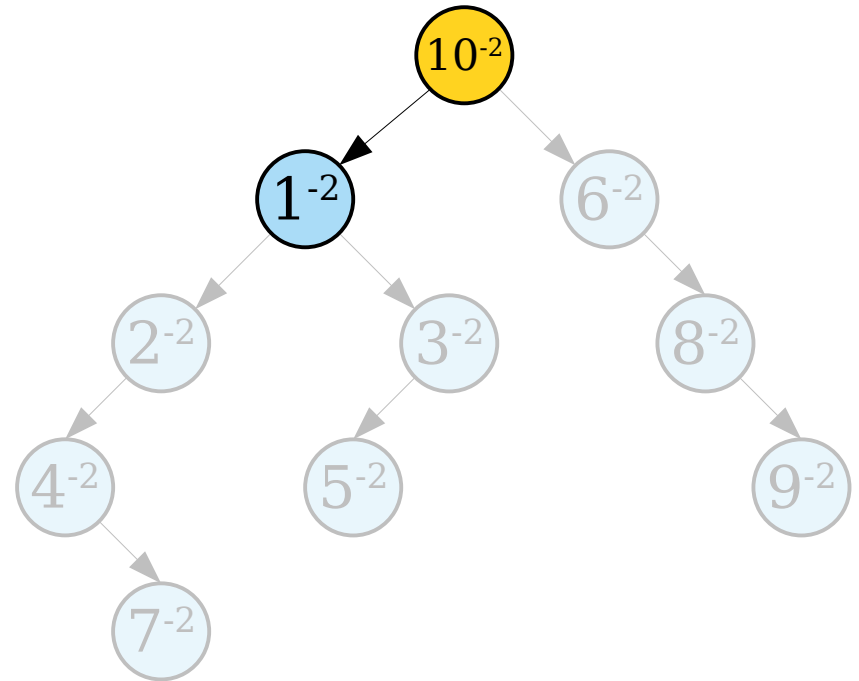
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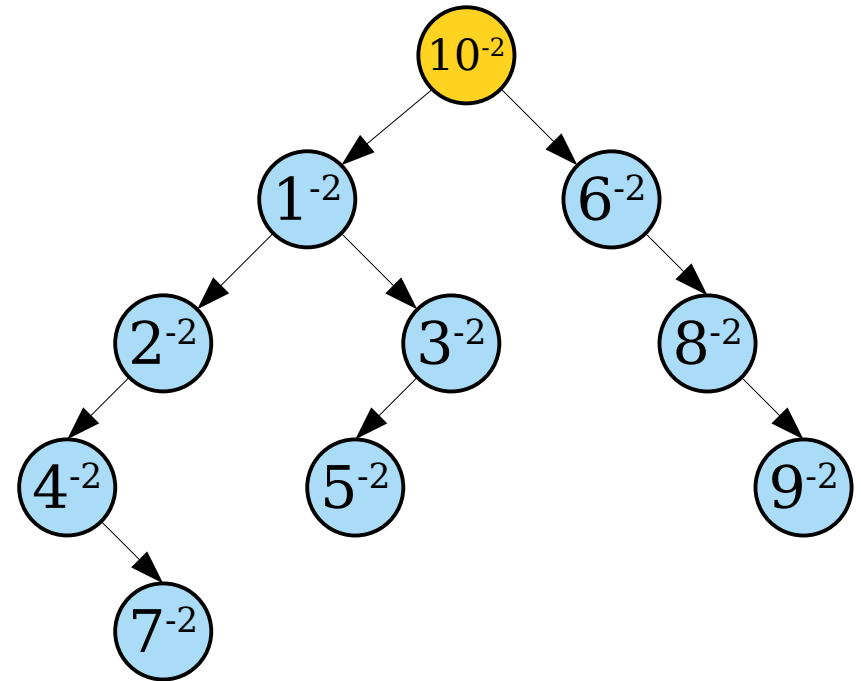
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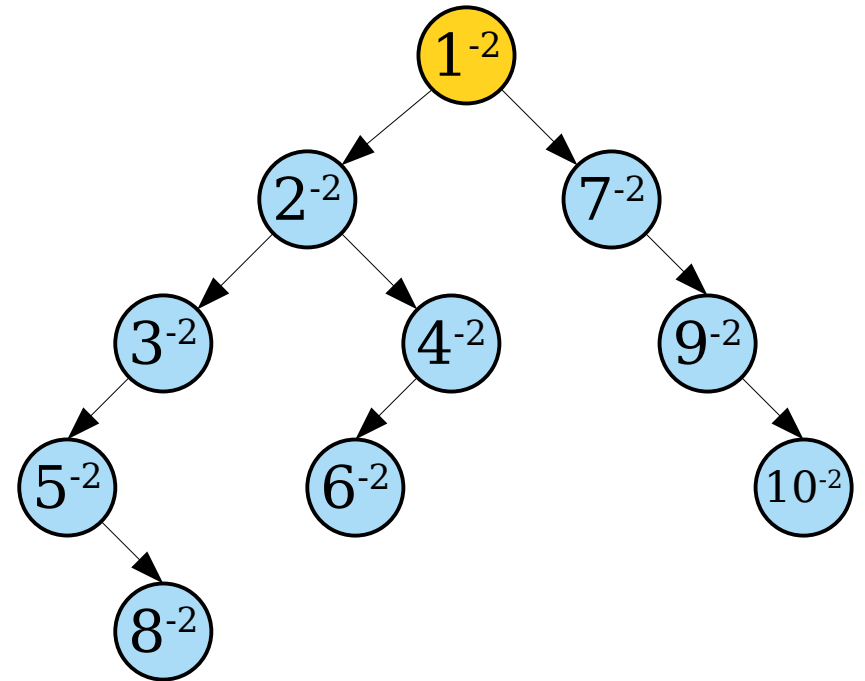
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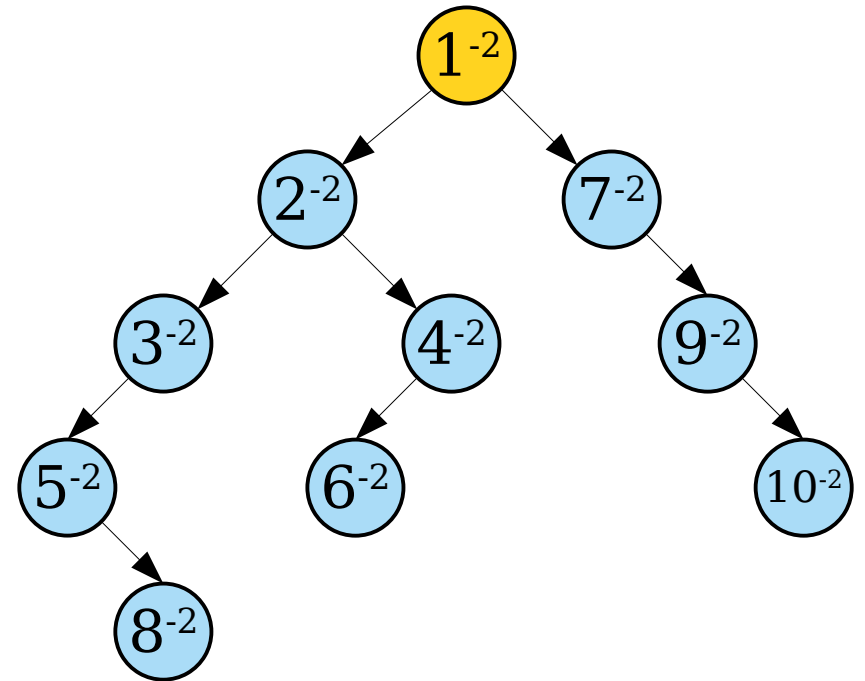
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$$\Phi = \sum_{i=1}^n \lg s(x_i).$$

Changing weights this way can't increase  $s(x_i)$  for any node, so  $\Phi$  can't increase after the splay.



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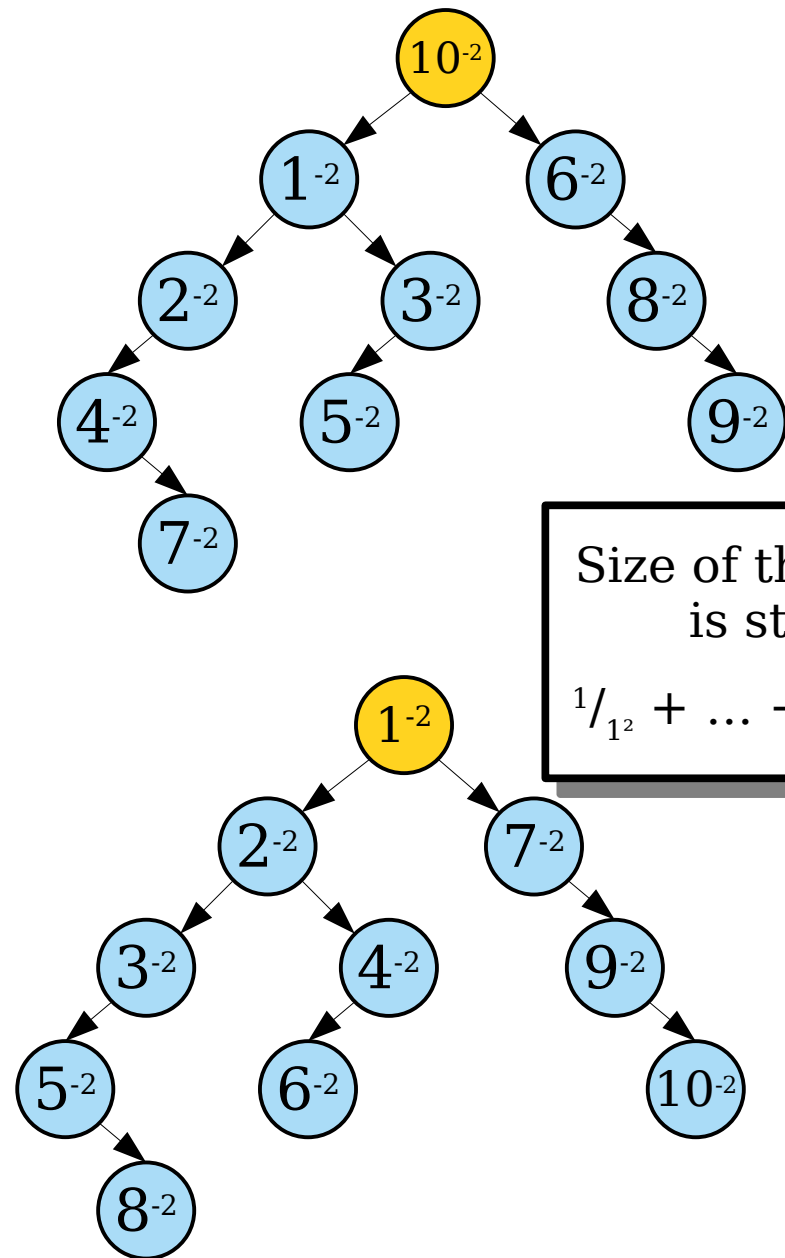
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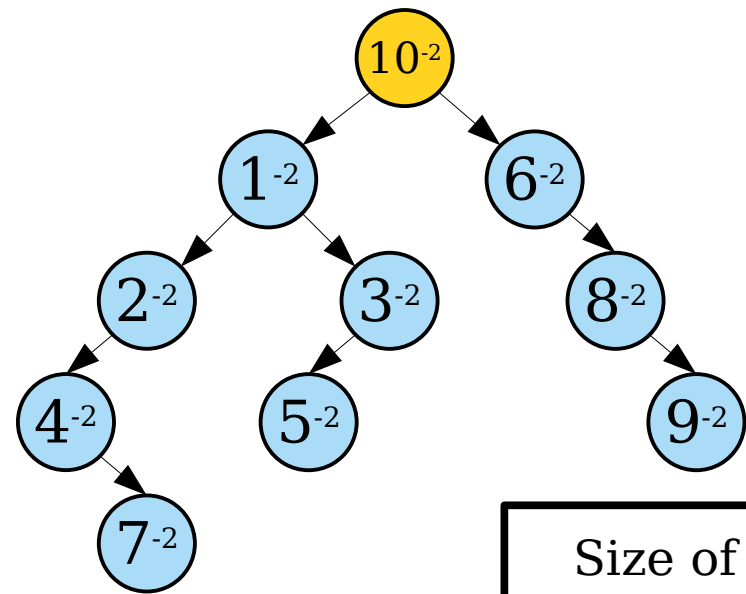
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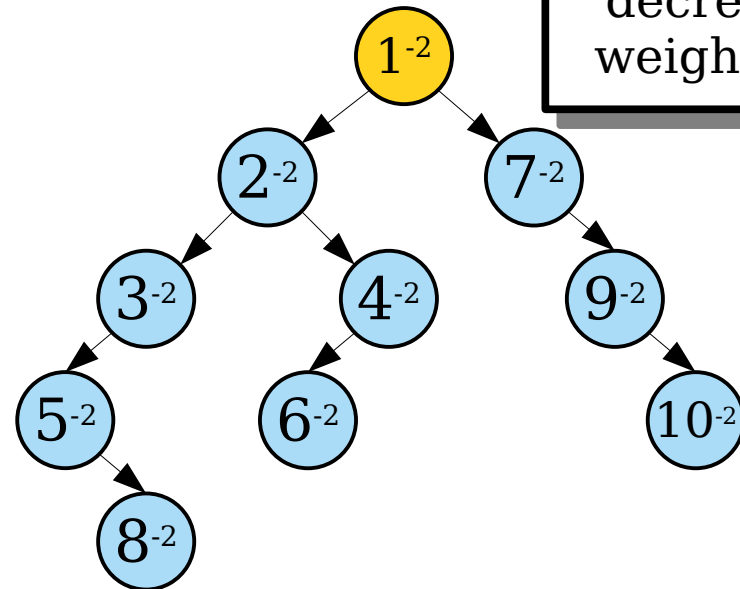
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Size of each other node decreases as weights drop.



**Lemma:** Using the sum-of-logs potential, the amortized cost of splaying a node with weight  $w_i$  is  $O(1 + \log W + \log (1/w_i))$ , where  $W$  is the sum of all the weights.



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So the amortized cost of each operation is still  $O(\log t_i)$ , even in the dynamic case!

Amortized Cost

=

Real Cost +  $\Delta\Phi$

Decreasing  $\Phi$  after the operation can only *reduce* the amortized cost.

**Lemma:** Using the sum-of-logs potential, the amortized cost of splaying a node with weight  $w_i$  is  $O(1 + \log W + \log (1/w_i))$ , where  $W$  is the sum of all the weights.

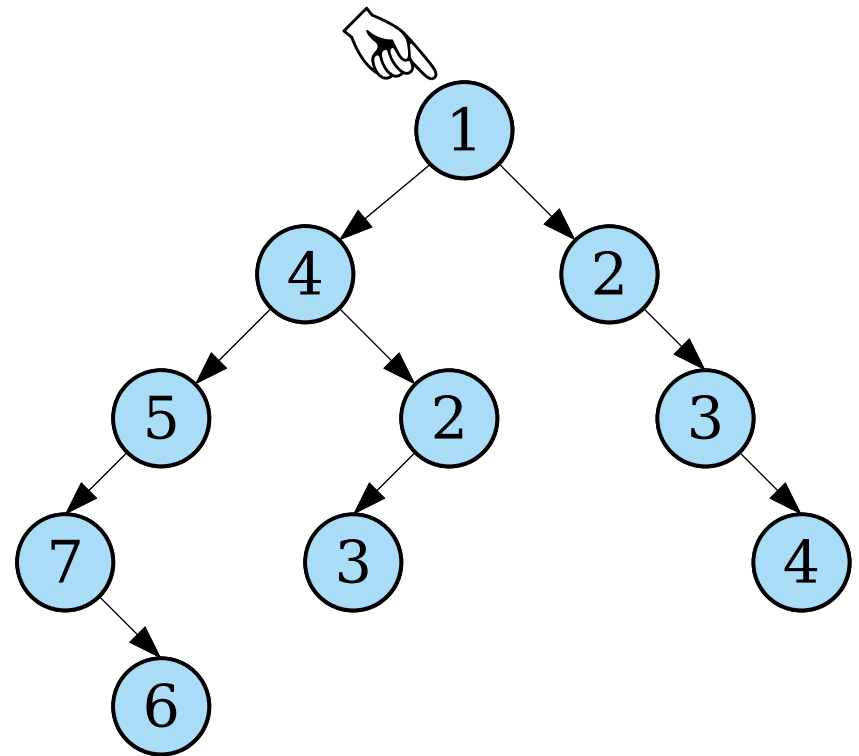
### **Dynamic Finger Property:**

Lookups take time  $O(\log \Delta)$ , where  $\Delta$  is the number of keys between the last key queried and the current key queried.

It doesn't immediately seem like we can use the theorem below, since the value of  $\Delta$  depends on what accesses have been done recently.

For now, let's set that aside and focus on one snapshot in time.

Each key  $x_i$  is annotated with its value  $\Delta_i$ , the rank difference to the last element. How do we pick weights?



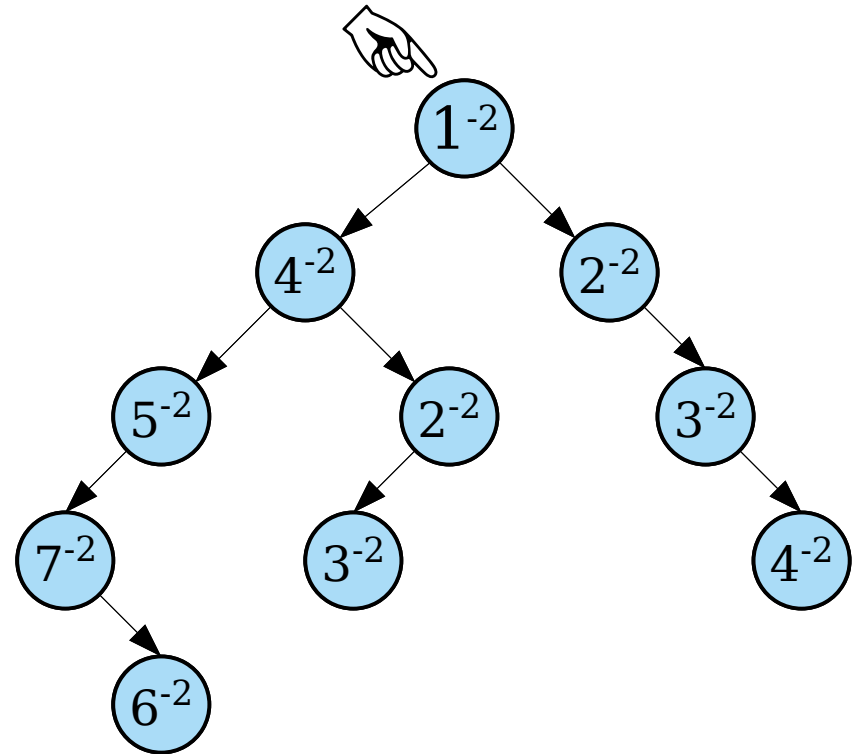
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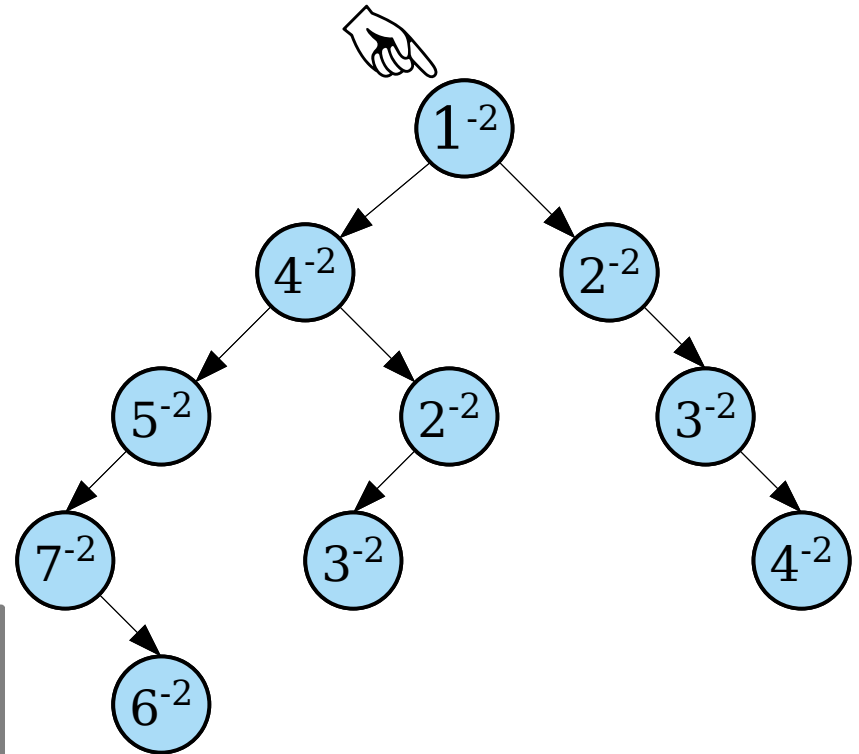
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$$W \leq \frac{2}{1^2} + \frac{2}{2^2} + \frac{2}{3^2} + \dots + \frac{2}{n^2} \\ = O(1)$$

There are at most two keys at distance  $k$  from the finger, one in each direction.



**Lemma:** Using the sum-of-logs potential, the amortized cost of splaying a node with weight  $w_i$  is  $O(1 + \log W + \log (1/w_i))$ , where  $W$  is the sum of all the weights.

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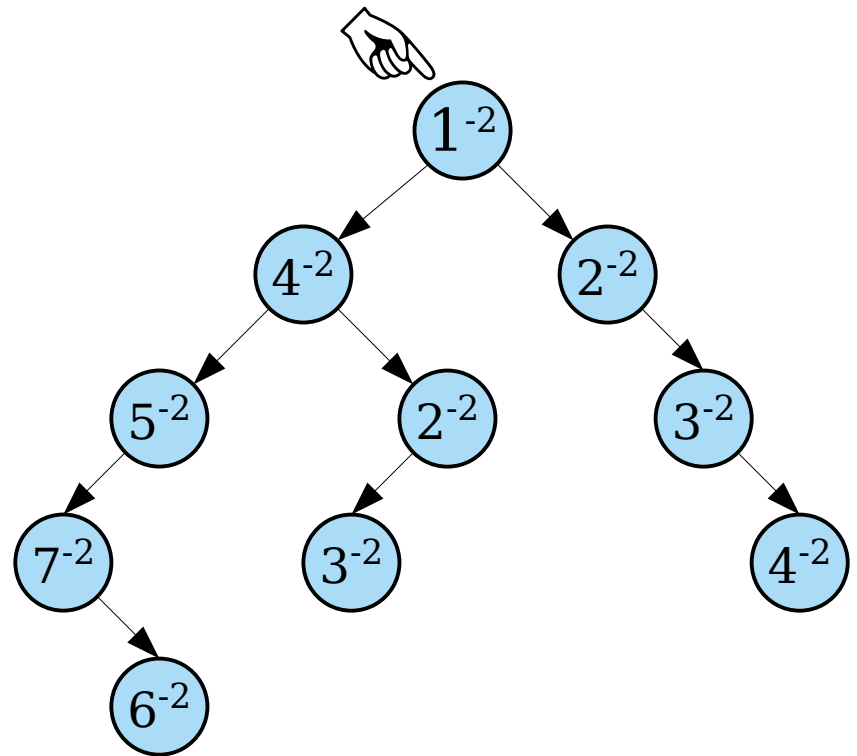
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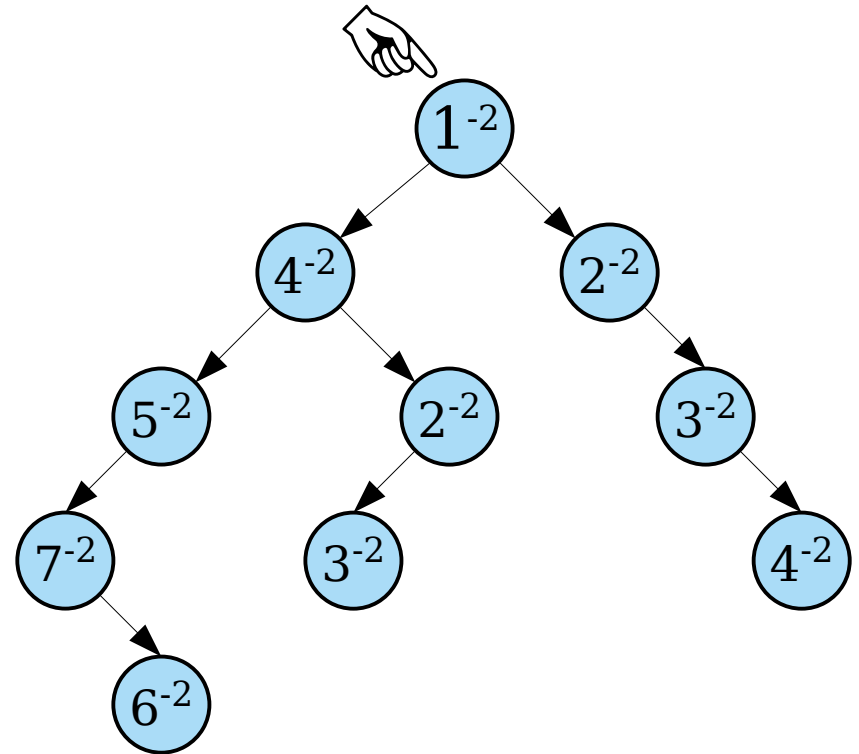
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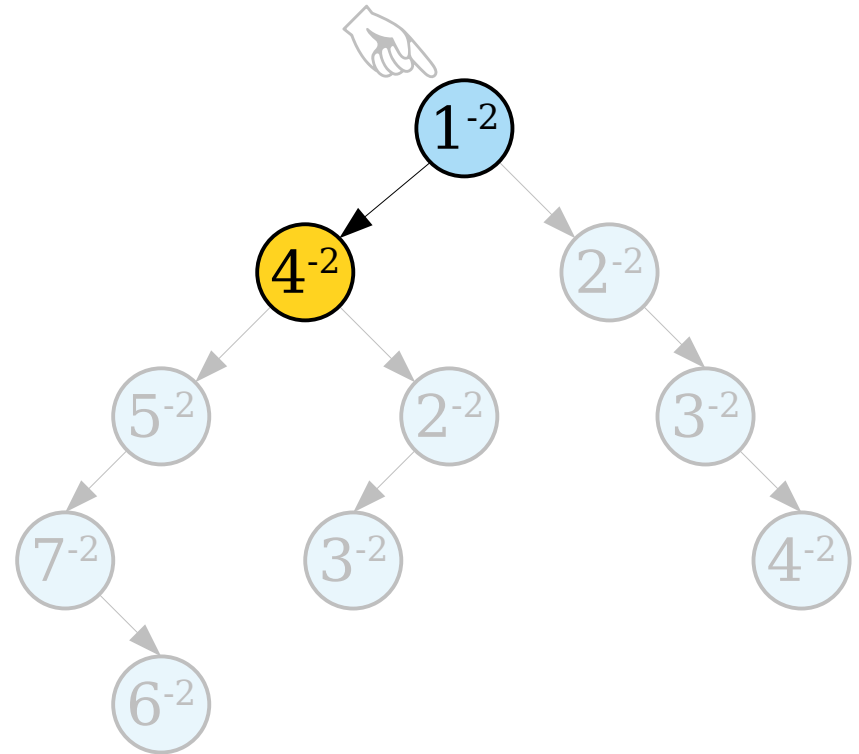
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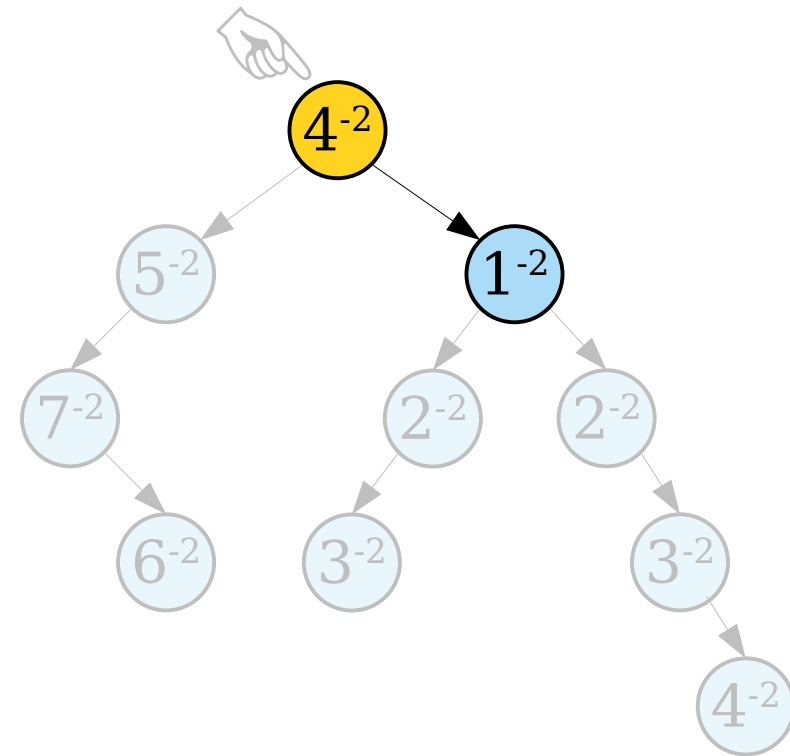
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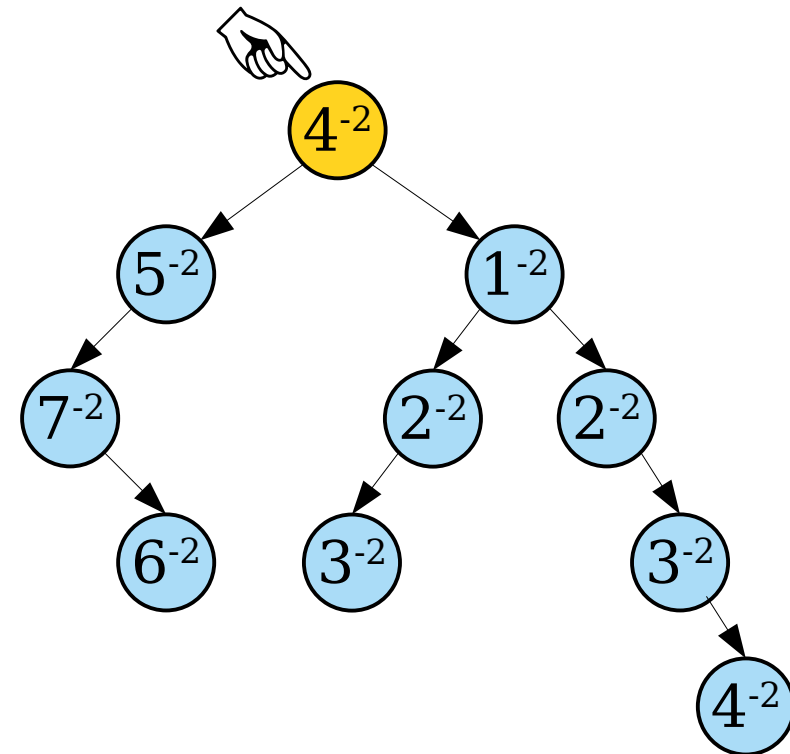


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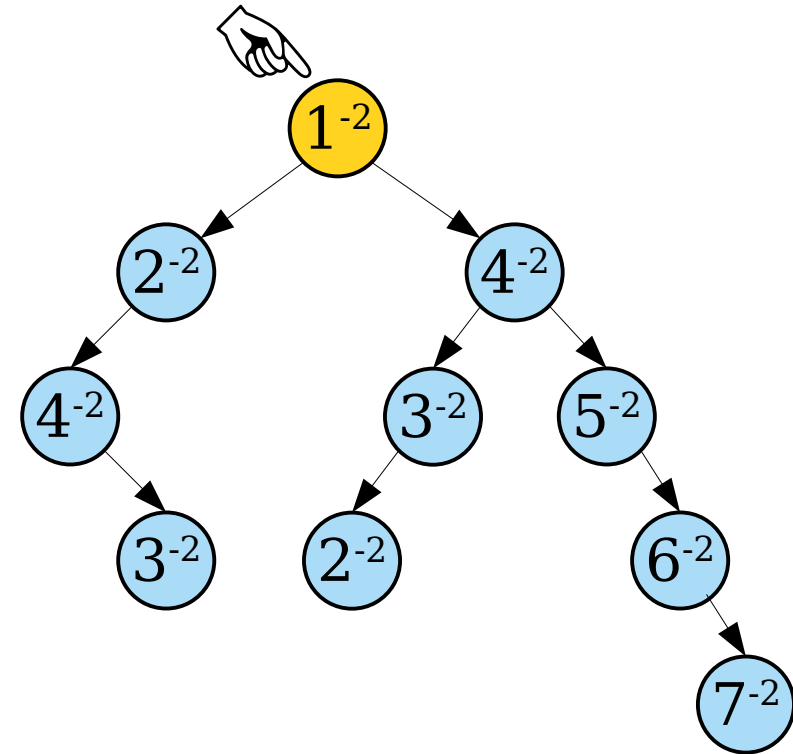
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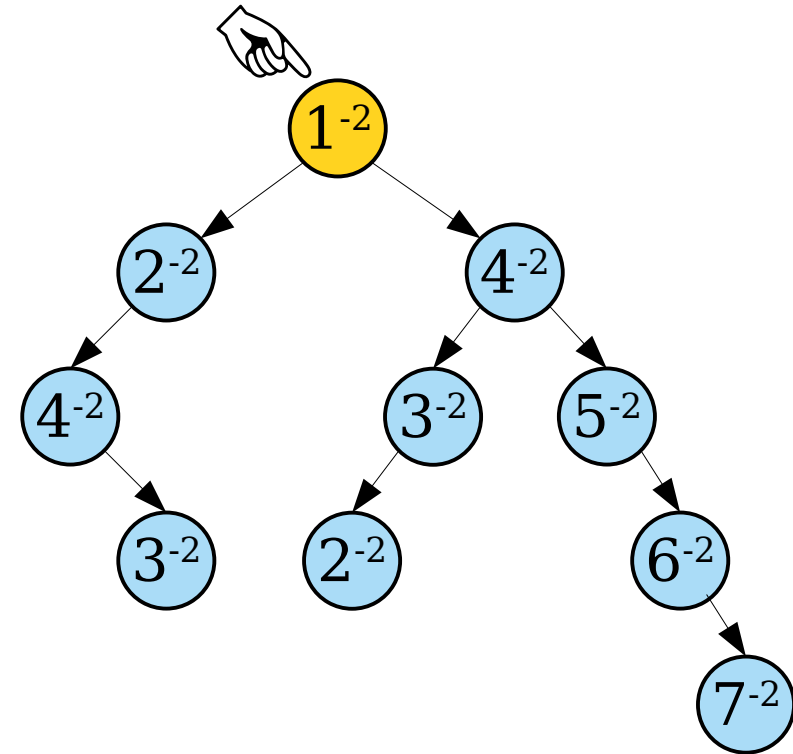


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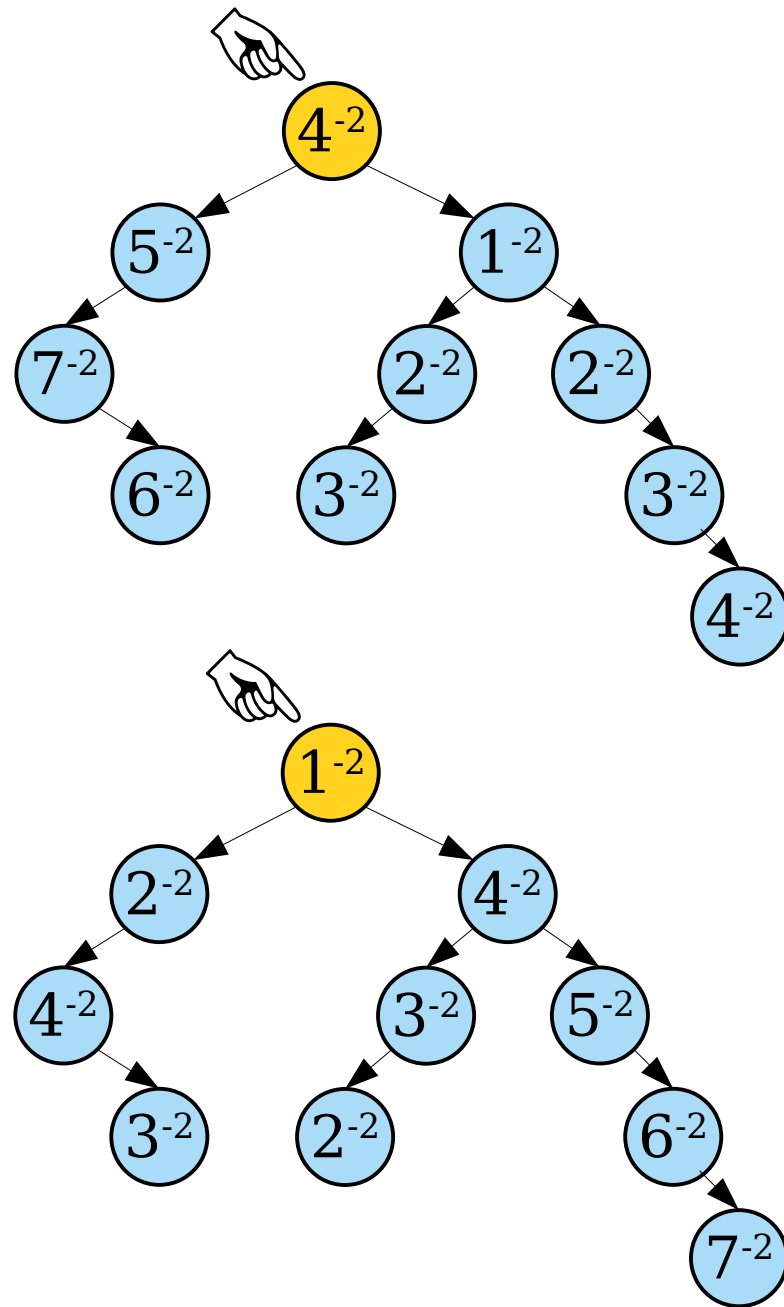


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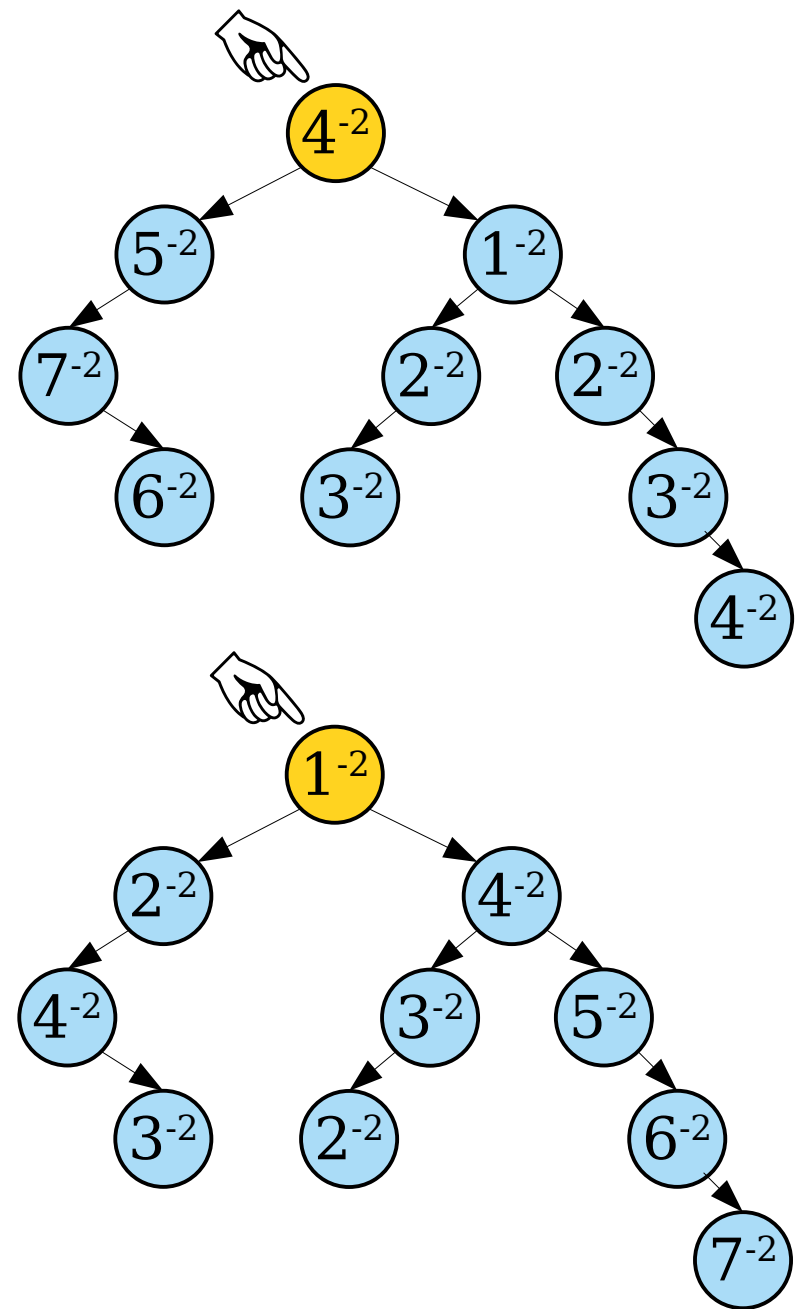
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However, we did just prove the **static finger property**: if you fix some key in advance and let  $\delta_i$  be the number of keys between  $x_i$  and that key, then lookups take time  $O(\log \delta_i)$ .



**Lemma:** Using the sum-of-logs potential, the amortized cost of splaying a node with weight  $w_i$  is  $O(1 + \log W + \log (1/w_i))$ , where  $W$  is the sum of all the weights.

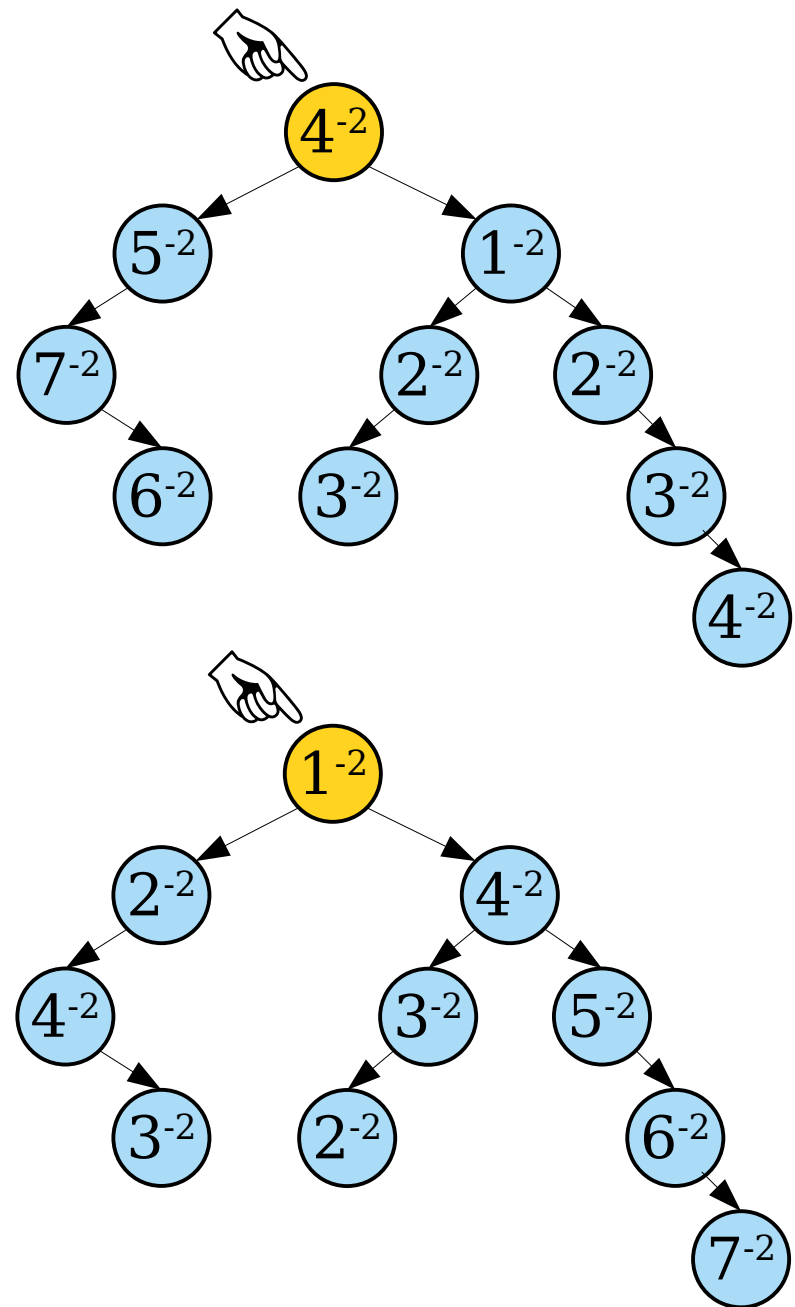
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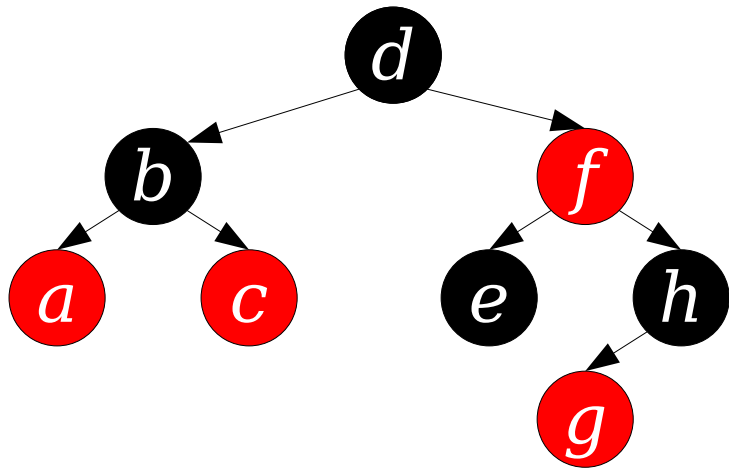
**Theorem:** Splay trees have the dynamic finger property.

**Proof:** 85 pages of analysis. See Cole et al, "On the Dynamic Finger Conjecture for Splay Trees."

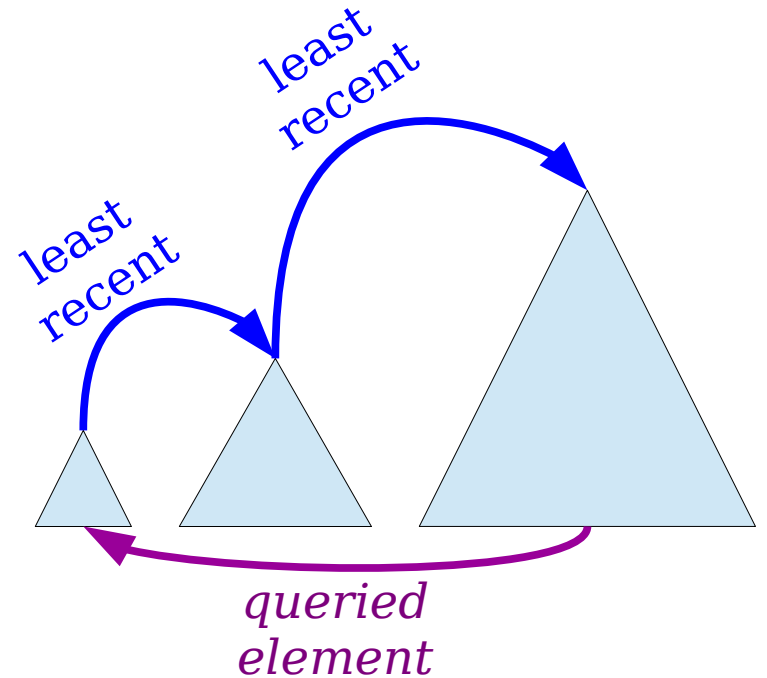
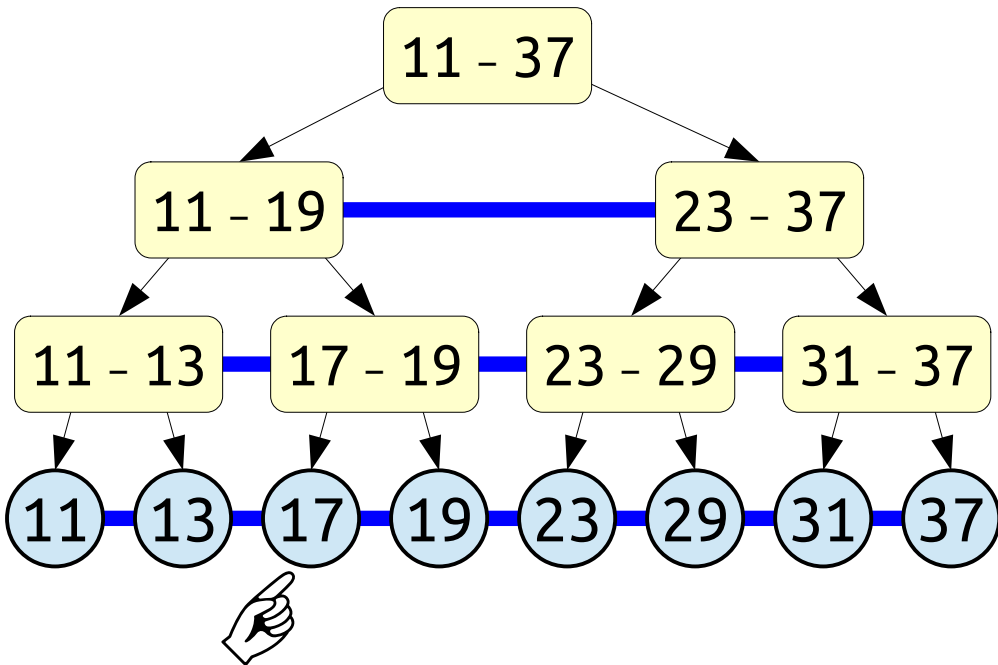
**Open Problem:** Find a simpler proof that splay trees have the dynamic finger property.



**Lemma:** Using the sum-of-logs potential, the amortized cost of splaying a node with weight  $w_i$  is  $O(1 + \log W + \log (1/w_i))$ , where  $W$  is the sum of all the weights.



Is all the creativity that goes into each of these structures captured by a single, simple binary search tree?



Just how fast *are* splay trees?

Pick any (long) sequence of operations. Pick any BST  $T$ , including one that, like a splay tree, is allowed to reshape itself.

***Dynamic Optimality Conjecture:***

**Cost of performing those operations on a splay tree**

**$\leq$**

**$O(1) \cdot$  Cost of performing those operations on  $T$**

Stated differently: no matter how clever you are with your BST design, you will never be able to beat a splay tree by more than a constant factor.

***This is an open problem! And it's a big one!***

Just how fast *are* splay trees?



So... if splay trees are so great, why aren't we using them everywhere instead of other tree structures?

1. Amortized versus worst-case bounds are not always acceptable in practice.
2. Poor support for concurrency, especially in lookup-heavy loads.
3. Slightly higher constant factors than some other trees, due to the number of memory writes per operation.

Many of drawbacks can be mitigated in practice, and we do see splay trees used fairly extensively in practice alongside red/black and B-trees.

***Excellent Idea:*** Code up splay trees and measure their performance!

Just how fast *are* splay trees?

- Worst-case efficiency (the *balance property*) isn't the only metric we can use to measure BST performance.
- Specialized data structures like weight-balanced trees, level-linked finger search trees, and Iacono's structure can be designed to meet these bounds.
- For a BST to have all these properties at once, it needs to be able to move nodes around.
- Rotate-to-root is a plausible but inefficient mechanism for reordering nodes.
- Splaying corrects for rotate-to-root by handling linear chains more intelligently.
- Splaying provides simple implementations of all common BST operations.
- By using a heavy/light decomposition, we can isolate the effects of poor tree shapes.
- Using a sum-of-logs potential allows us to amortize away heavy edges.
- Splay trees have the balance property, entropy property, dynamic finger property, and working set property.
- It's an open problem in data structure theory whether it's possible to improve upon splay trees in an amortized sense.

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To Summarize...

# Next Time

- ***Dynamic Connectivity***
  - Answering questions about graphs as those graphs change.
- ***Disjoint-Set Forests***
  - Solving a special case of dynamic connectivity.
- ***The Ackermann Inverse Function***
  - A shockingly slowly-growing function with a more fearsome reputation than it deserves.