Suffix and LCP Arrays

Recap from Last Time

Suffix Trees

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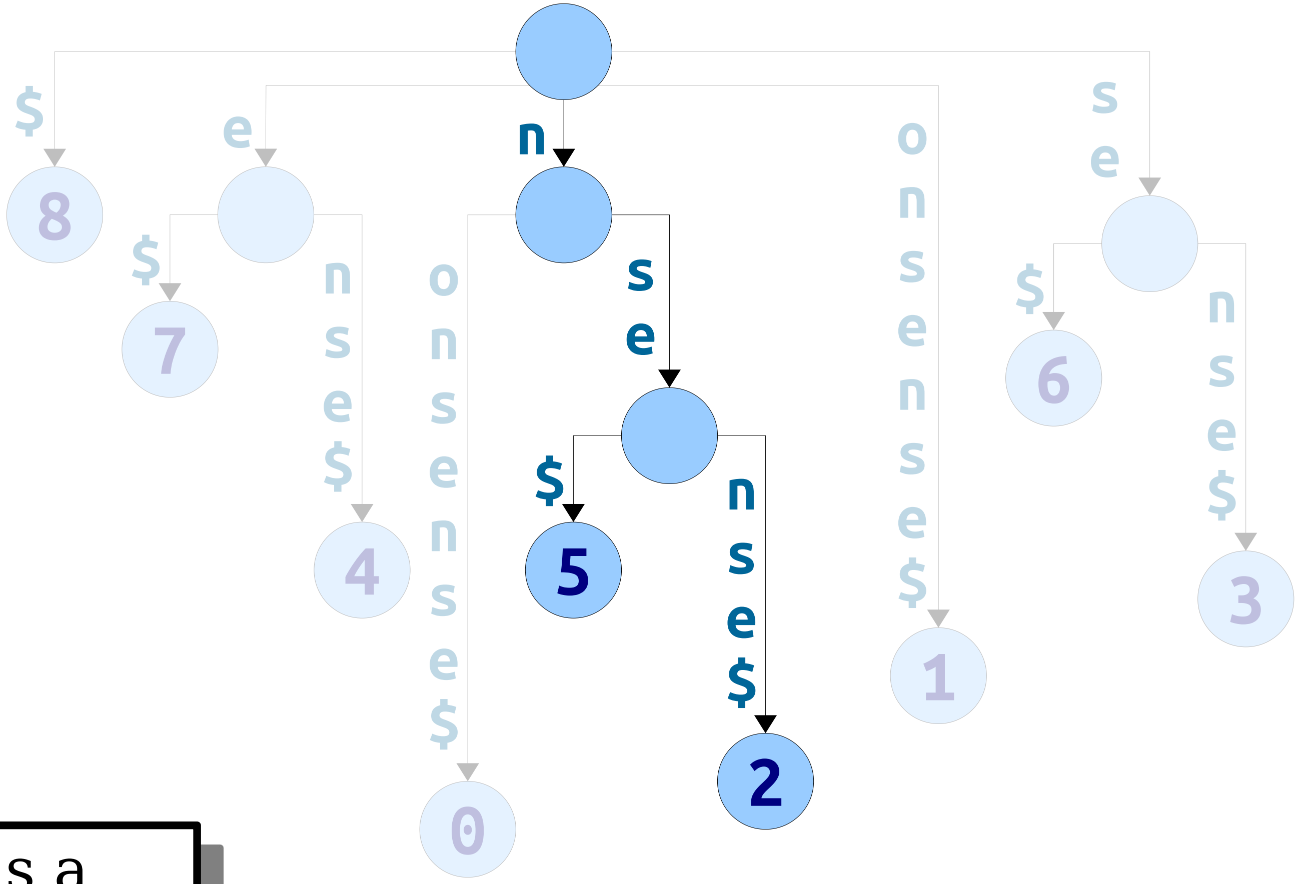


nonsense$
012345678

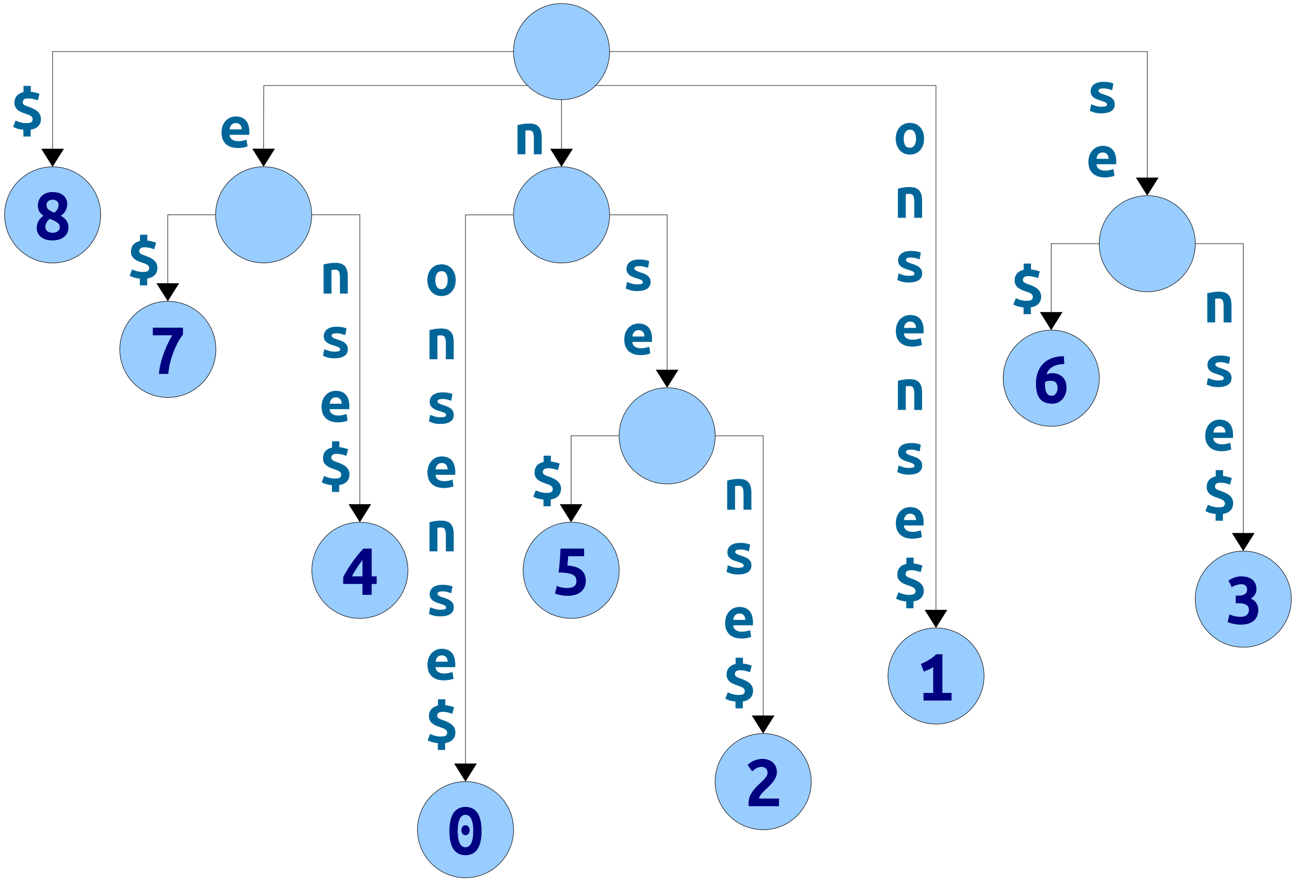
Theorem: *w* is a substring of *x* if and only if *w* is a prefix of a suffix of *x*.

New Stuff!

Representing Suffix Trees

Representing a Suffix Tree

- We know that a suffix tree has O(***m***) nodes, where ***m*** is the number of characters in the input string.
- This means that there are O(***m***) edges.
- ***Question:*** Why can't we immediately claim that the space usage of the suffix tree is O(***m***)?



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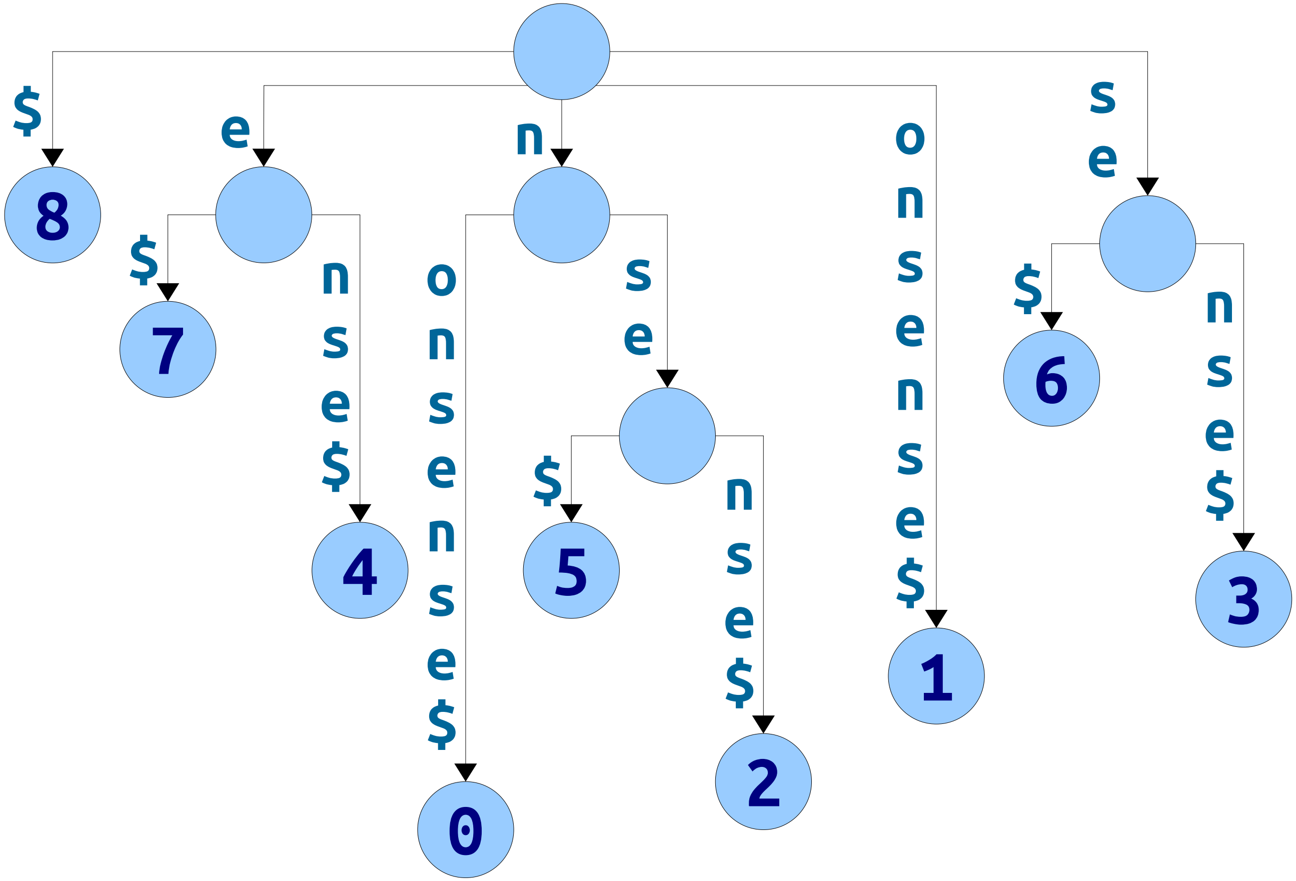
Representing a Suffix Tree

- ***Claim:*** Writing out all suffixes of a string of length m requires $\Theta(m^2)$ characters.
- ***Proof idea:*** Those suffixes have length $1 + 2 + \ldots + (m+1)$, factoring in the special **$** character.
- ***Problem:*** It is indeed possible to build a suffix tree with $\Theta(m^2)$ total letters on the edges.



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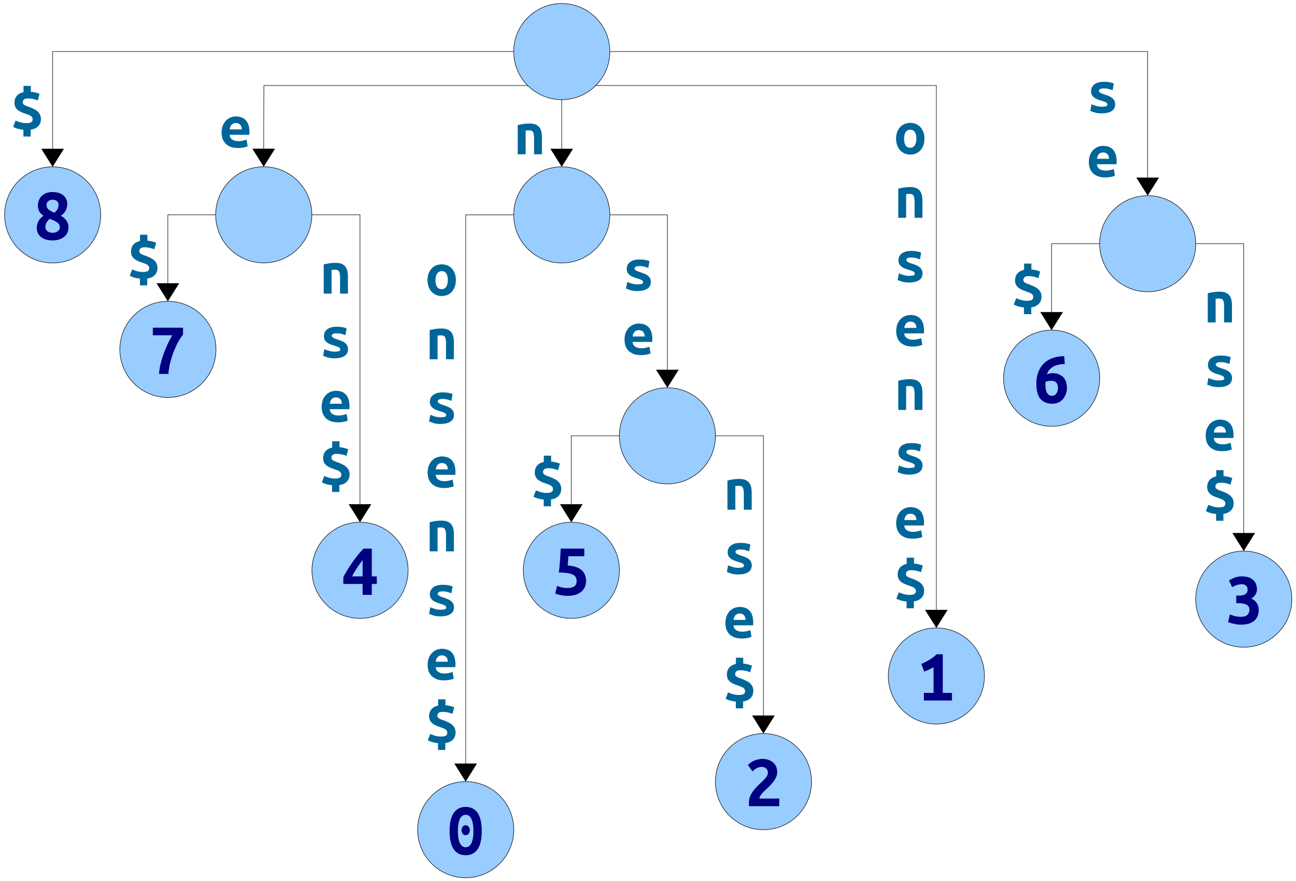
Representing a Suffix Tree

- By being clever with our representation, we can guarantee that a suffix tree uses only Θ(***m***) space, regardless of the input string.
- ***Observation:*** Each edge is labeled with a substring of the original input string.
- ***Idea:*** Don’t actually write out the labels on the edges. Just write down the start and end index!



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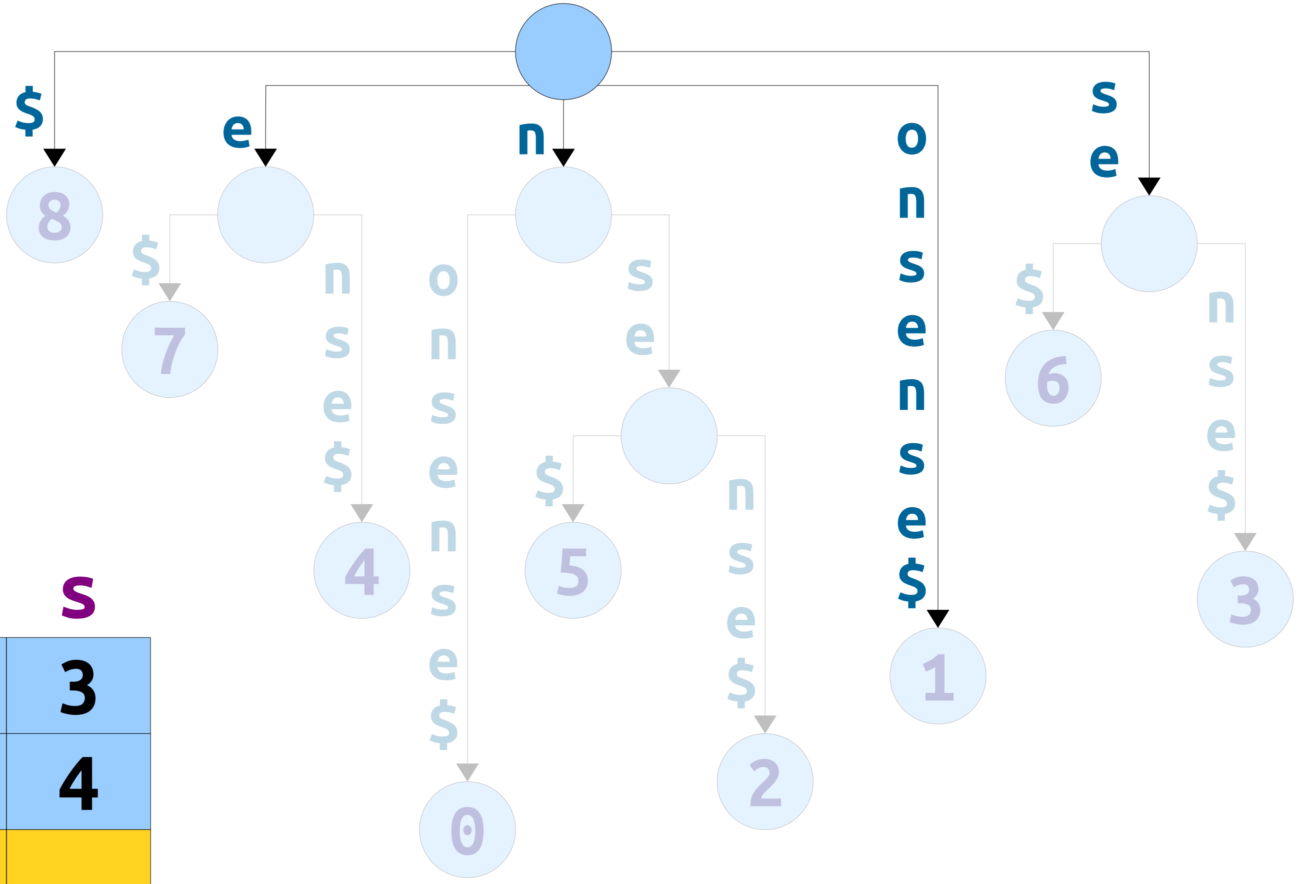
Representing a Suffix Tree

	$	e	n	o	s
start	8	4	0	1	3
end	8	4	0	8	4
child					



nonsense$
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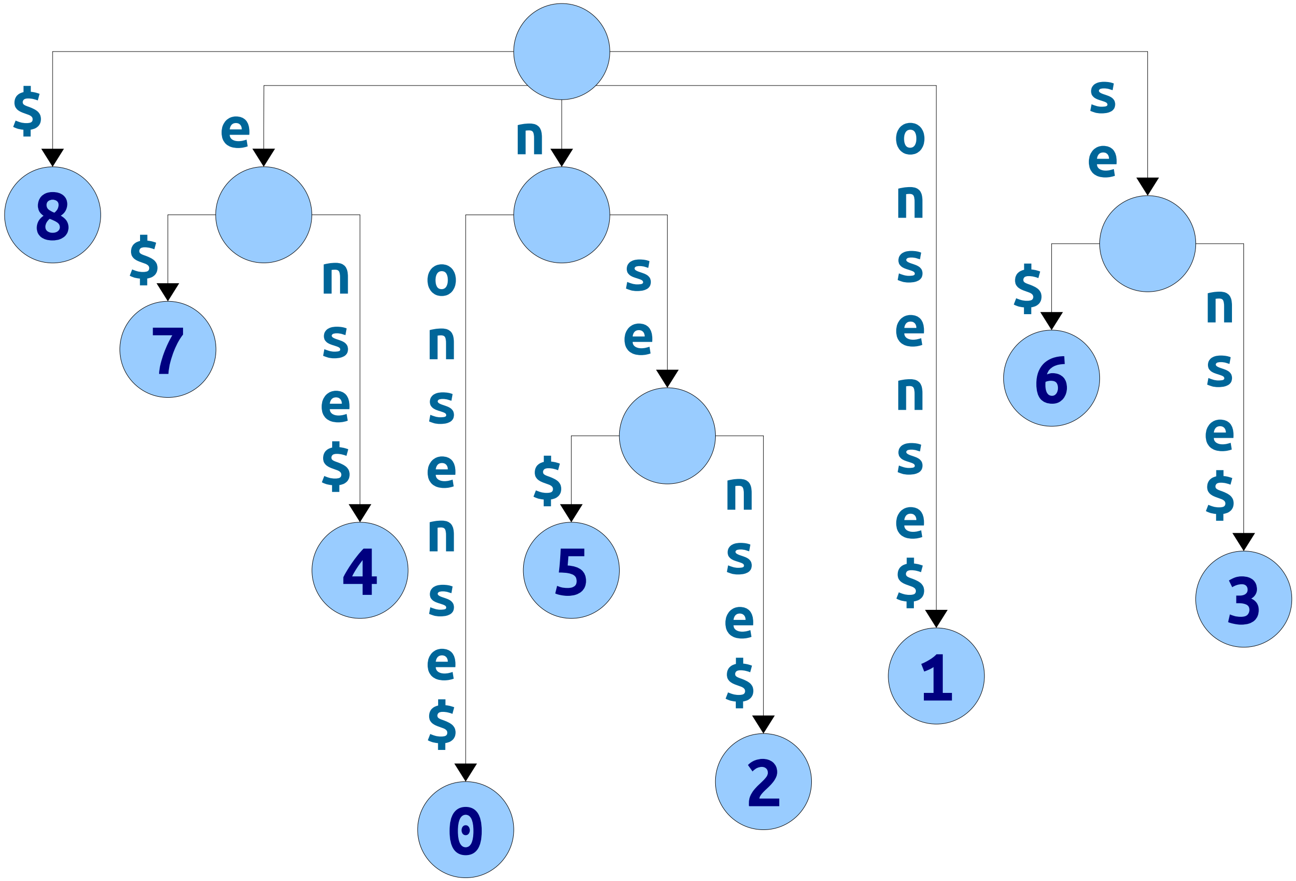
Representing a Suffix Tree

- Space usage required for a suffix tree:
  - O(***m***) space for all the nodes.
  - O(***m***) space for a copy of the original string.
  - O(***m***) space for the edges.
- Total space: O(***m***).



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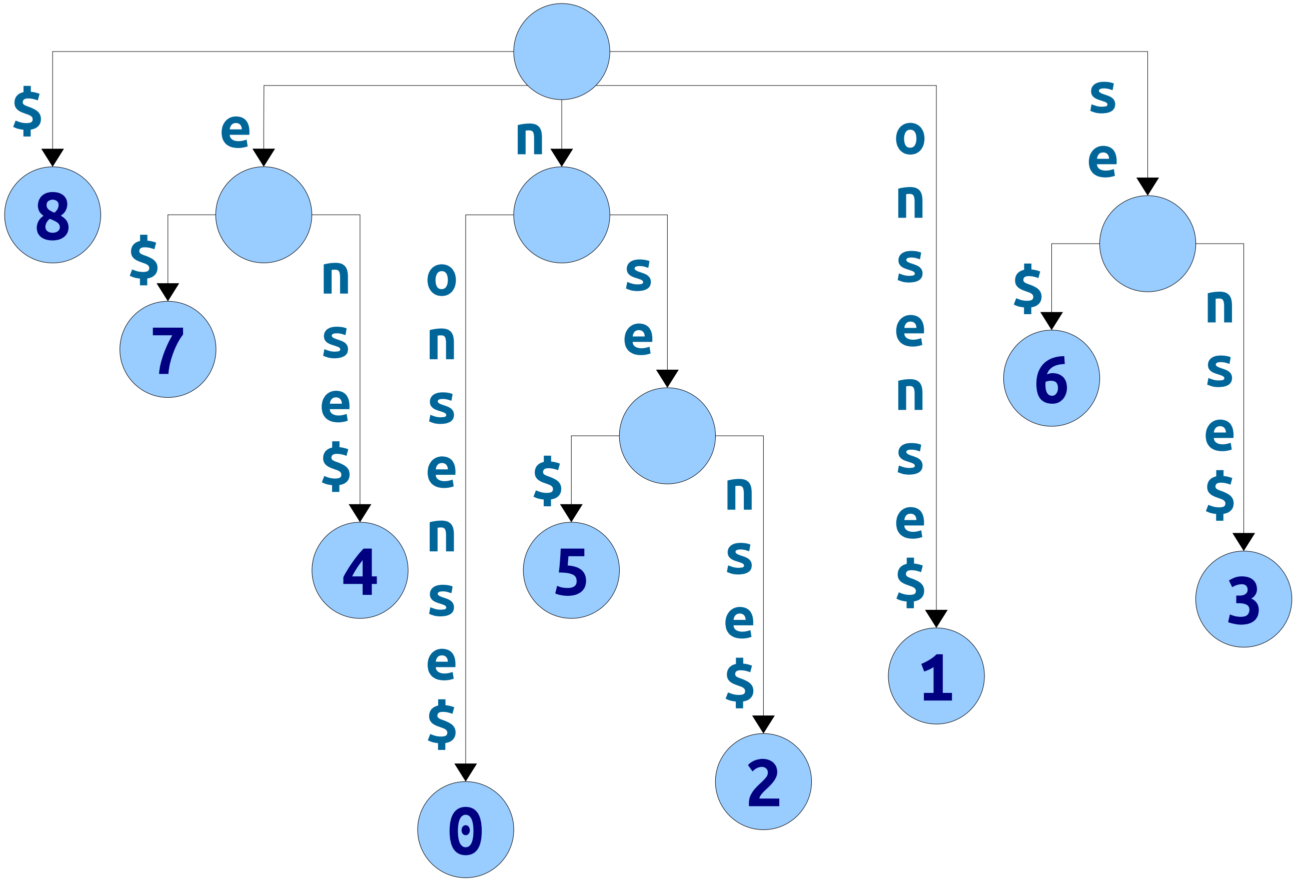
Suffix Tree Space Usage

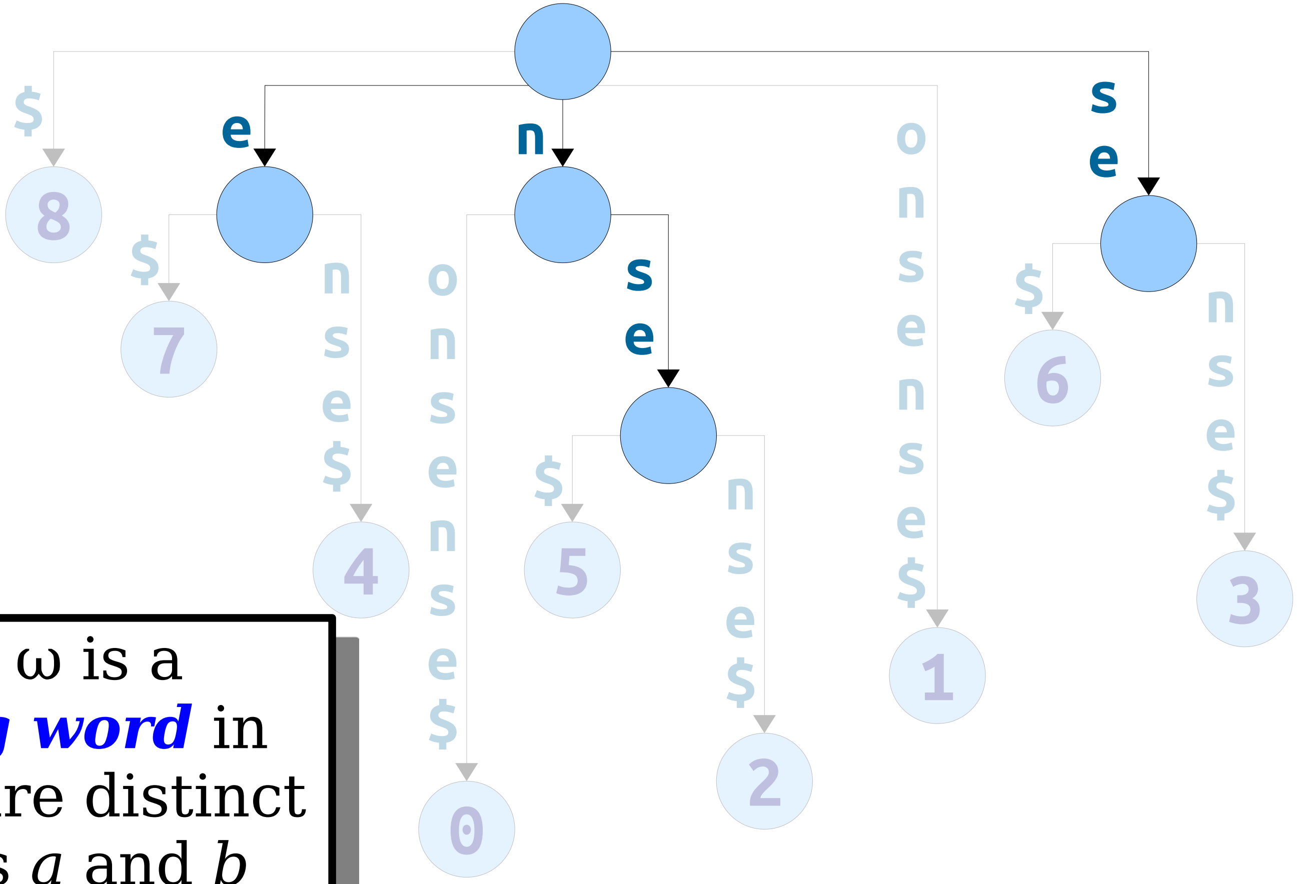
- Suffix tree edges take up a *lot* of space.
  - Two machine words per edge to denote the range of characters visited.
  - One machine word per edge for the pointer itself.
  - Number of edges ranges from m to $2m$ – 1, so this is between $3m$ and $6m$ machine words for the whole string!
- Example: a human genome is about three billion characters long.
  - With clever techniques, that can be packed into about 800MB.
  - On a 32-bit machine, the suffix tree needs about 48GB – too big to fit into memory!
  - On a 64-bit machine, the suffix tree needs about 96GB – way more than a typical machine can hold!

Key Question: Can we get the benefits of a suffix tree without the space penalty?

What is it about suffix trees that make them so useful algorithmically?

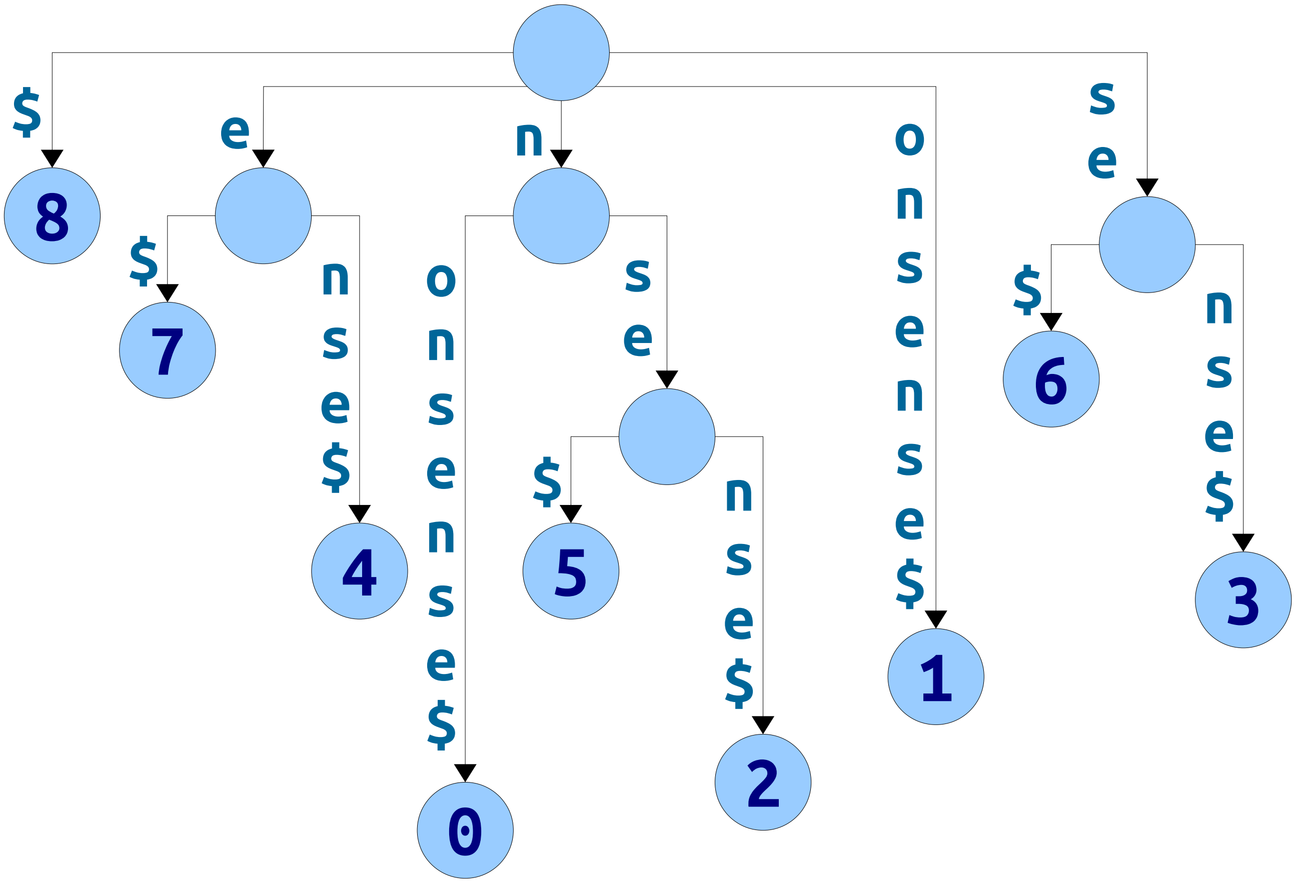


Theorem: There is a node labeled ω in a suffix tree for *T*
if and only if
ω is a suffix of *T*$ or ω is a branching word in *T*$.



A string ω is a ***branching word*** in *T*$ if there are distinct characters *a* and *b* where ω*a* and ω*b* are substrings of *T*$.

Theorem: There is a node labeled ω in a suffix tree for *T* *if and only if* ω is a suffix of *T*$ or ω is a branching word in *T*$.



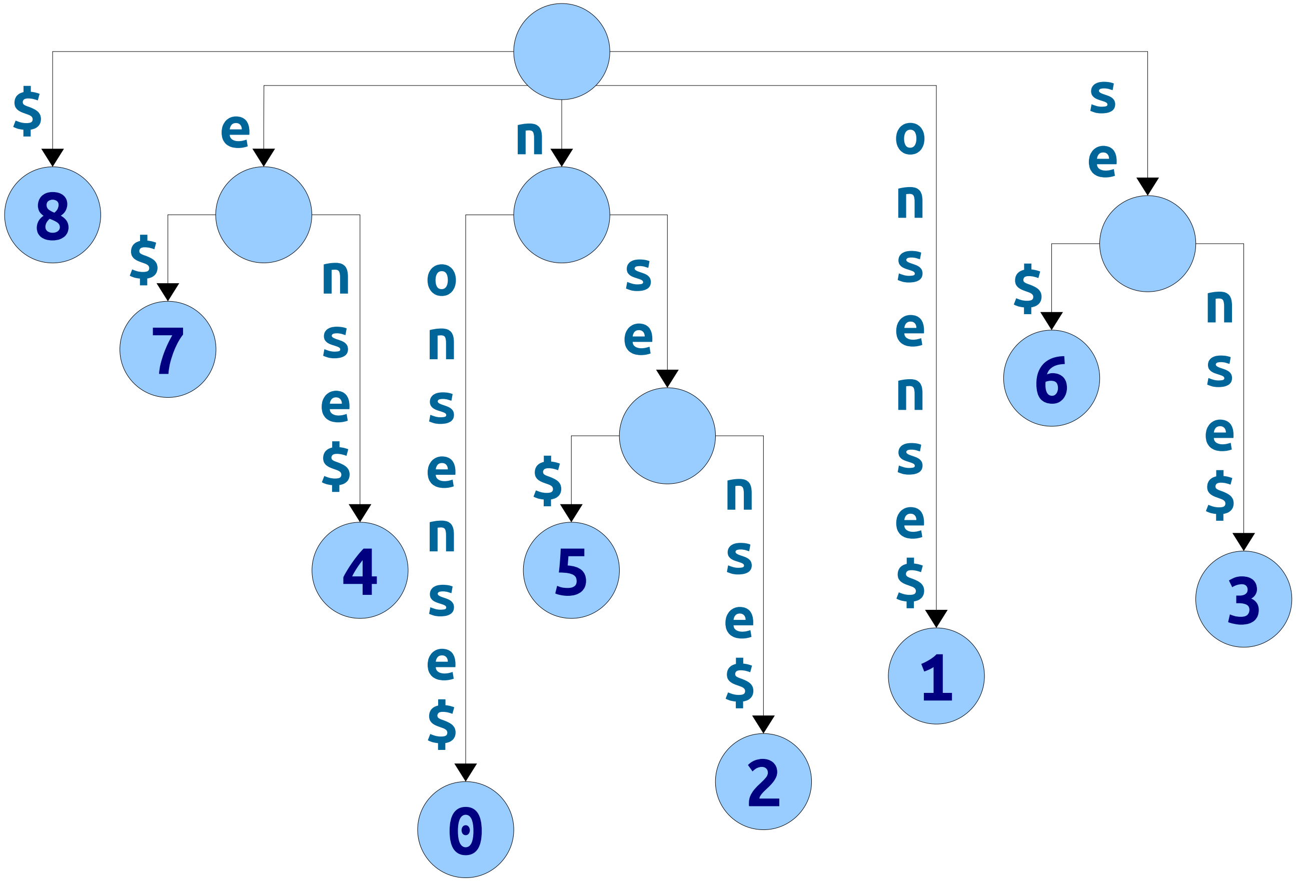
Key Intuition: The efficiency in a suffix tree is largely due to

1. keeping the suffixes in sorted order, and
2. exposing branching words.

Where We’re Going

- Today, we’ll see two data structures that encode much of the same information as suffix trees, but in much less space.
  - The ***suffix array*** stores information about the ordering of the suffixes of a string.
  - The ***LCP array*** stores information about the branching words of a string.
- Together, they’ll provide algorithms that match or are comparable to the time bounds from last time.

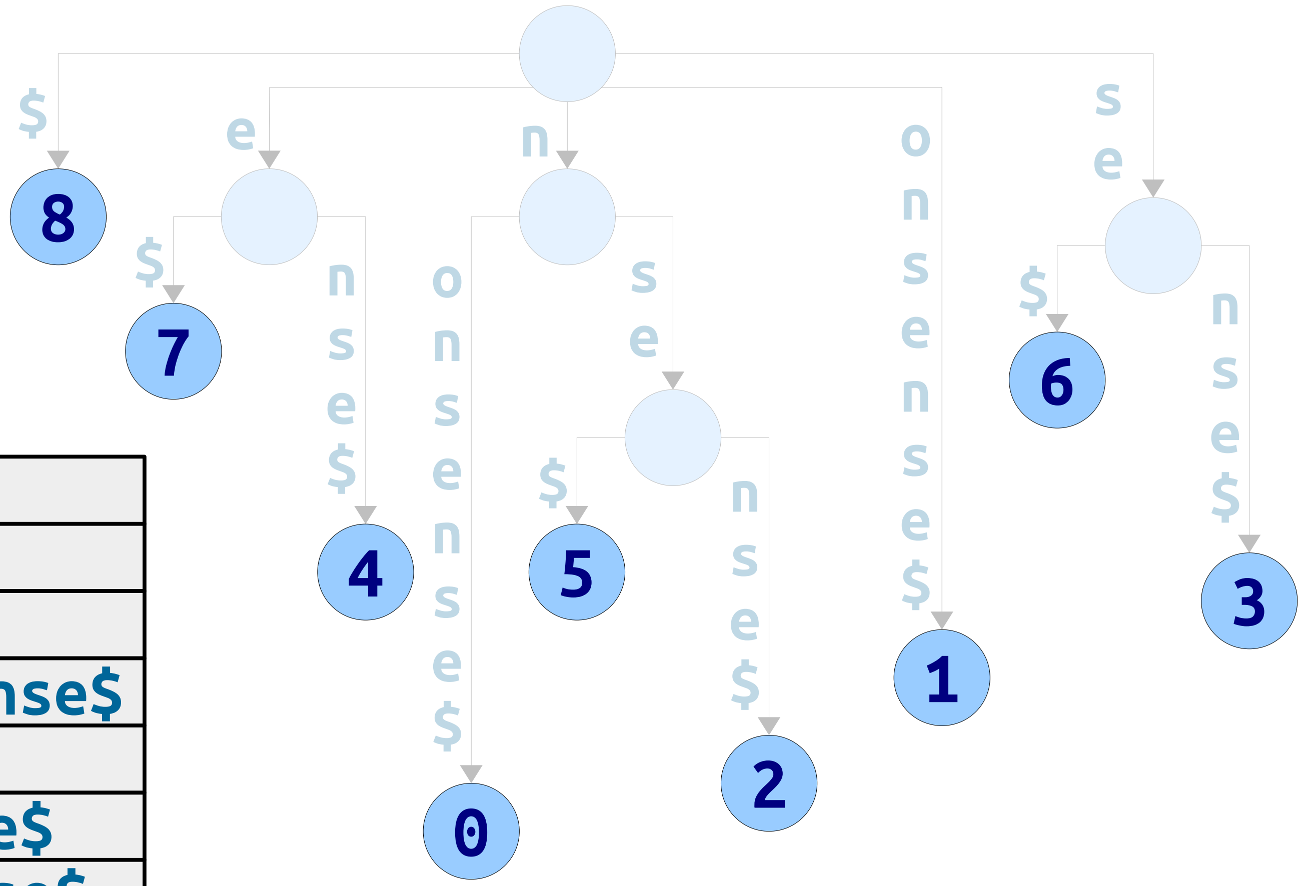
Suffix Arrays



Theorem: There is a node labeled ω in a suffix tree for *T*

if and only if

ω is a suffix of *T*\$ or ω is a branching word in *T*\$.



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Theorem: There is a node labeled ω in a suffix tree for *T*
if and only if
ω is a suffix of *T*$ or ω is a branching word in *T*$.

Suffix Arrays

- A ***suffix array*** for a string *T* is a sorted array of the suffixes of the string *T*$.
- Suffix arrays distill out just the first component of suffix trees: they store suffixes in sorted order.
- ***Non-obvious fact:*** Suffix arrays can be built in time O(***m***). We can cover this later in the quarter if you're interested.

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Suffix Arrays

- The way we've drawn suffix arrays is terribly space-inefficient.
    - It always uses space $\Theta(\boldsymbol{m}^2)$, since that's how many total characters occur in all suffixes.
- Can we do better?

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Suffix Arrays

- We reduced the space usage of suffix trees by representing substrings, implicitly, as ranges within the original string.
- ***Idea:*** Don’t store the suffixes themselves. Just store the starting positions of the suffixes.
- Space: Θ(***m***), and with only one machine word used per character of input.

14
13
0
6
11
4
2
8
1
7
10
12
5
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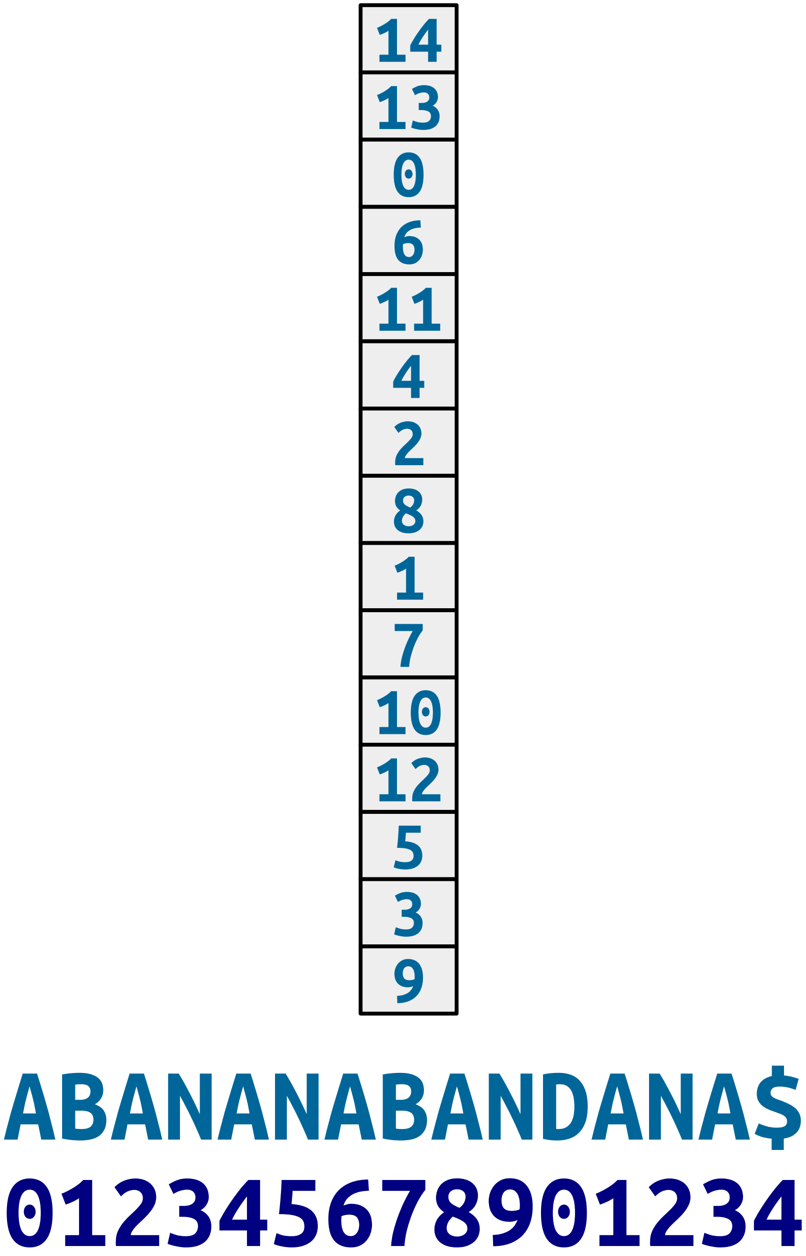
9

ABANANABANDANA$
012345678901234

Suffix Arrays

- Although the picture to the right is how we’d represent the suffix array in memory, for this lecture we’ll draw things out the longer way.
- This is just to build intuition; we wouldn’t actually do that in practice.



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ABANANABANDANA$
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Using Suffix Arrays

- Last time, we saw how to find all instances of a pattern ***P*** in a text ***T*** using suffix *trees*.
- How could we do that with suffix *arrays*?

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Using Suffix Arrays

- ***Reminder:*** Our text string ***T*** has length ***m***. Our pattern string ***P*** has length ***n***.
- ***Claim:*** With a suffix array, we can determine whether ***P*** appears in ***T*** in time O(***n*** log ***m***).

How?

Answer at

https://pollev.com/cs166spr23

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Using Suffix Arrays

- ***Reminder:*** Our text string ***T*** has length ***m***. Our pattern string ***P*** has length ***n***.
- ***Claim:*** With a suffix array, we can determine whether ***P*** appears in ***T*** in time O(***n*** log ***m***).
  - Binary search has O(log ***m***) rounds.
  - Each probe takes time O(***n***).
- This bound can be made tight. *(How?)*
- Figure that ***m*** is often much bigger than ***n***, so this is a huge win over a raw scan.

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ANAN

Using Suffix Arrays

- ***Claim:*** With a suffix array, we can find all matches of a pattern ***P*** in ***T*** in time $O(\boldsymbol{n} \log \boldsymbol{m} + \boldsymbol{z})$, where ***z*** is the number of matches.
- ***Idea:*** Binary search can be used to find a range of values equal to some key. Adapt that idea to find all suffixes beginning with the same prefix.

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NDANA$

ABANANABANDANA$

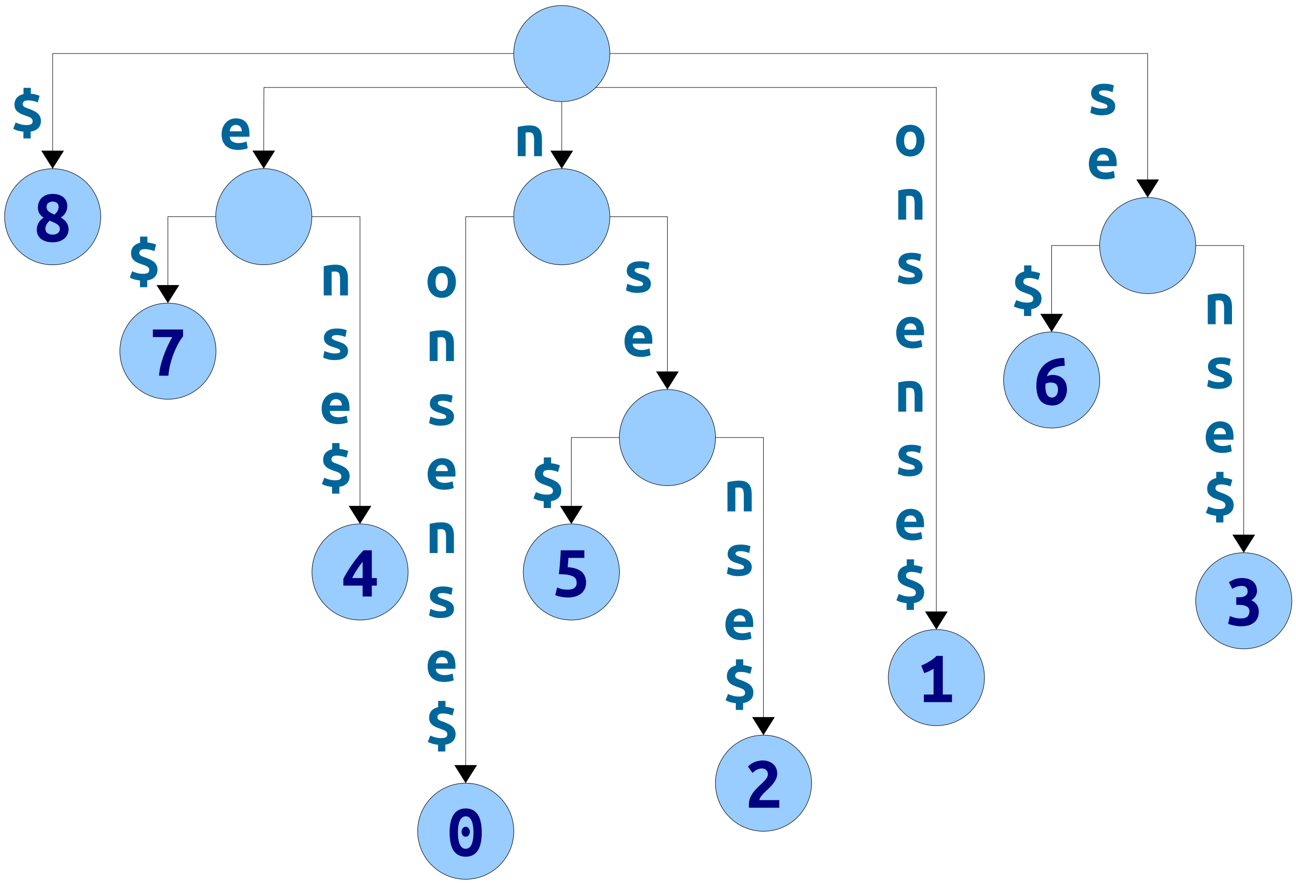
NA

The Story So Far

- Suffix arrays store all the suffixes of a string in sorted order.
- They provide an

  $$\langle O(\boldsymbol{m}), O(\boldsymbol{n} \log \boldsymbol{m} + \boldsymbol{z}) \rangle$$

  solution to the substring search problem.
- ***Intuition:*** Suffix trees are valuable in large part because they just keep the suffixes sorted.
- What else are suffix trees doing?



nonsense$
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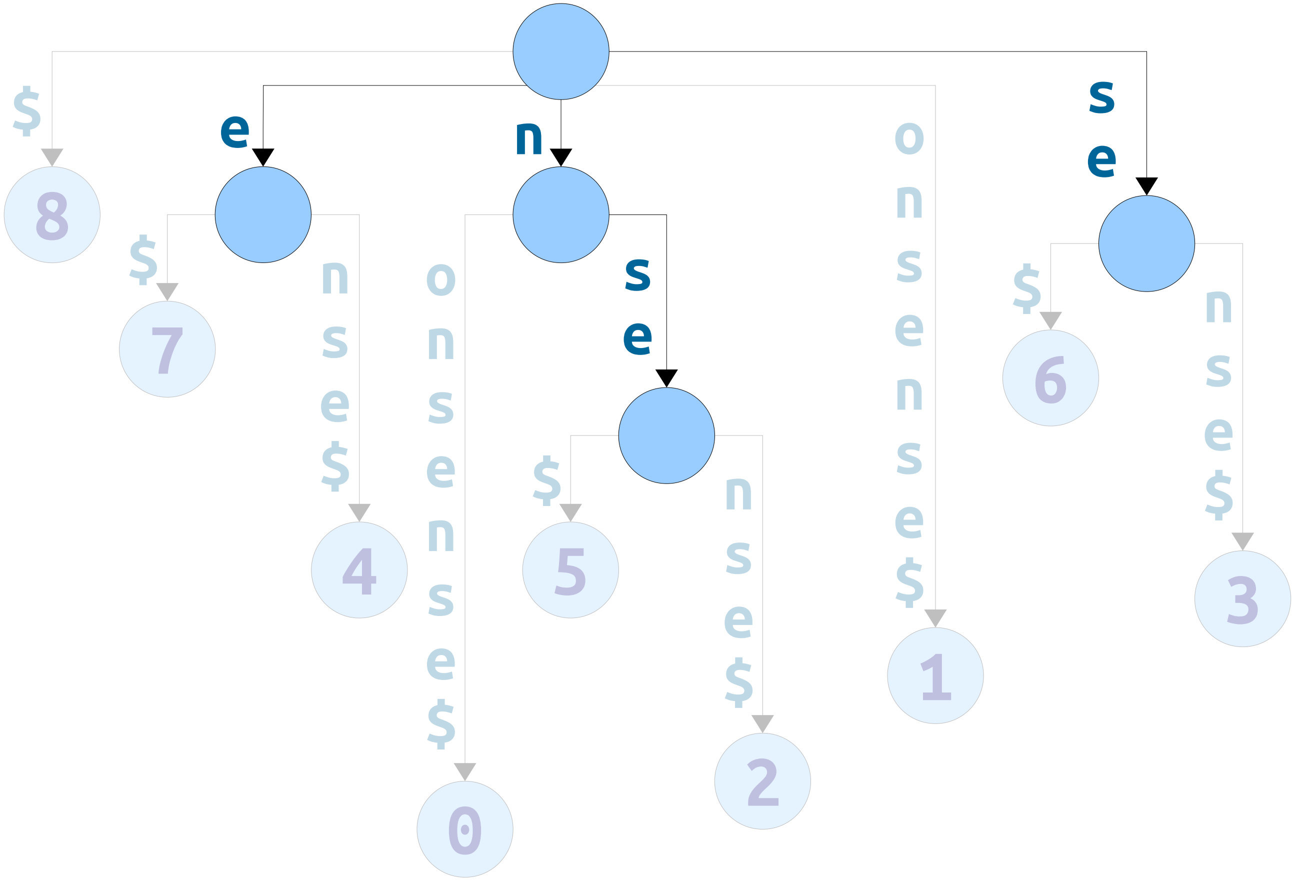
Theorem: There is a node labeled ω in a suffix tree for *T*

if and only if

ω is a suffix of *T*$ or ω is a branching word in *T*$.



nonsense$
012345678

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if and only if
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Branching Words

- ***Recall:*** If T is a string, then ω is a ***branching word*** in $T\$$ if there are characters $a \neq b$ such that ωa and ωb are substrings of $T\$$.

ABANANABANDANA$

Branching Words

- ***Recall:*** If T is a string, then ω is a ***branching word*** in $T\$$ if there are characters $a \neq b$ such that ωa and ωb are substrings of $T\$$.

Although `ABA` is a repeated substring, it is not a branching word because all appearances are followed by `N`.

ABANANABANDANA$

Branching Words

- ***Recall:*** If T is a string, then ω is a ***branching word*** in $T\$$ if there are characters $a \neq b$ such that ωa and ωb are substrings of $T\$$.

The substring ANANA only appears once, so it's not a branching word.

AB**ANANA**BANDANA$

Branching Words

- Notice that, by sorting suffixes, we've made it easier to spot branching words.
- Specifically, all suffixes starting with a branching word will be adjacent in the suffix array.
- The branching word will be the ***longest common prefix*** (or ***LCP***) of those adjacent suffixes.

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ABANDANA$
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NANABANDANA$
NDANA$

ABANANABANDANA$

Branching Words

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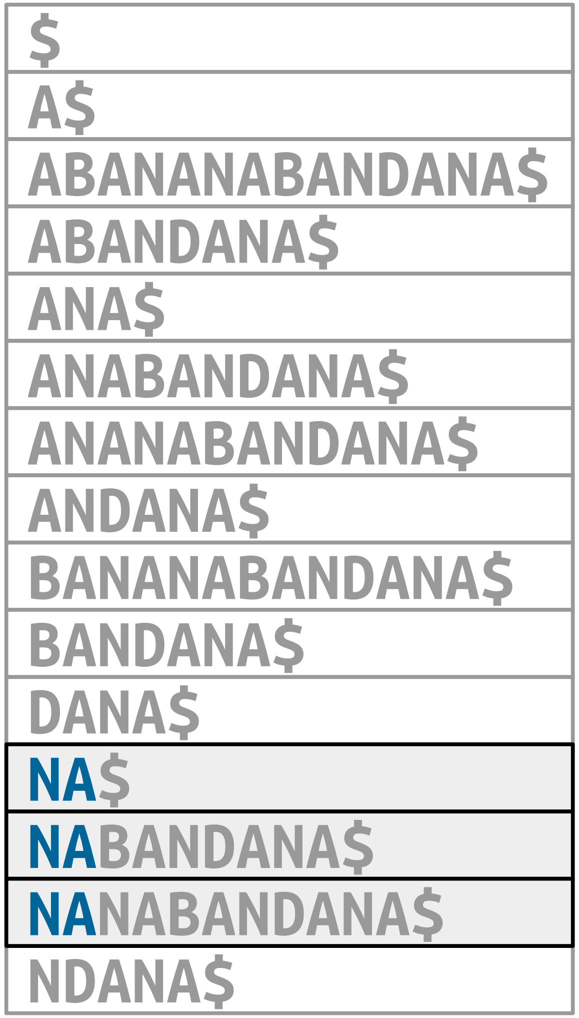

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NANABANDANA$
NDANA$

ABANANABANDANA$

Branching Words

- ***Theorem:*** A string ω is a branching word in string *T*$ if and only if it's the longest common prefix of two adjacent suffixes in *T*'s suffix array.
- ***Proof idea:*** If ω is the longest common prefix of two adjacent suffixes, let *a* and *b* be the characters immediately following ω in those two suffixes. Then ω*a* and ω*b* are substrings of *T*$.

  If ω is branching, choose the lexicographically smallest *a* and *b* making the definition work. Then the last suffix starting with ω*a* and the first suffix starting with ω*b* are adjacent in the suffix array. ■

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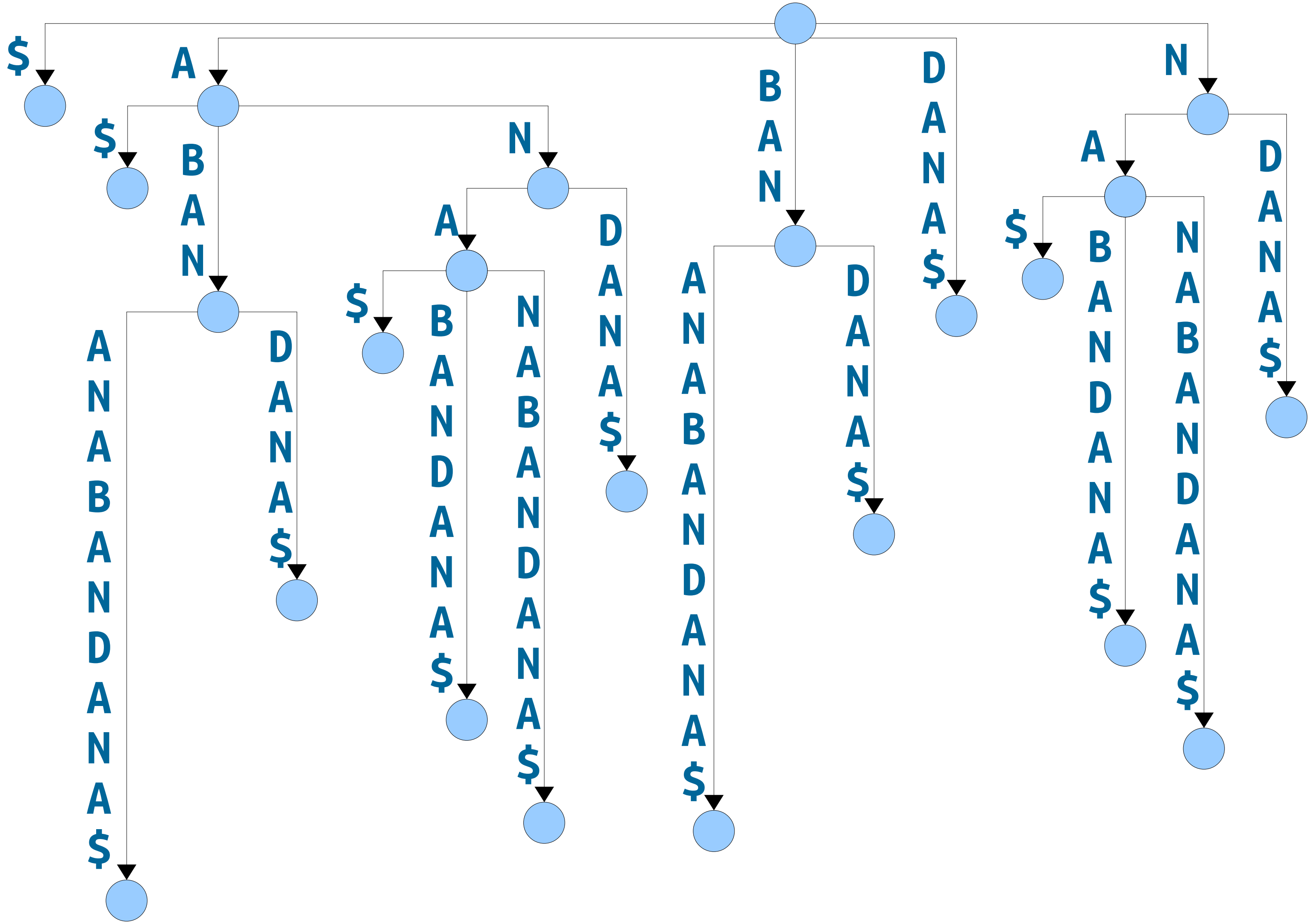

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NDANA$

ABANANABANDANA$

ω is an internal node in the suffix tree for T

if and only if

ω is a branching word in T$

if and only if

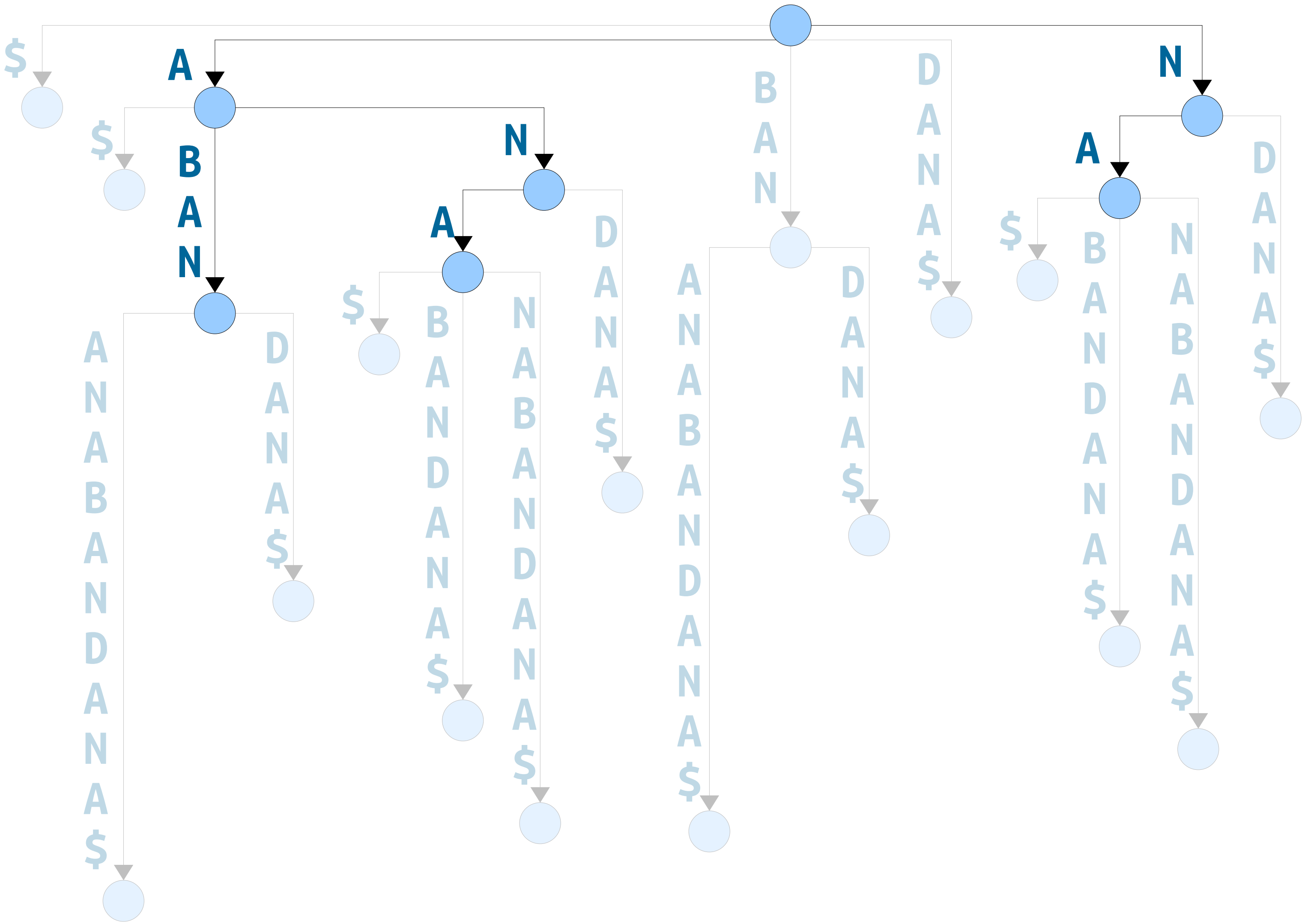
ω is the LCP of two adjacent suffixes in the suffix array for T

Key Intuition: Adjacent suffixes with long shared prefixes correspond to subtrees of the suffix tree.

Harnessing this Connection

Longest Repeated Substring

- Last time, we saw how to solve the longest repeated substring problem by using suffix trees.
- ***Algorithm:*** Find the internal node in the suffix tree with the longest label.
- ***Question:*** Can we do this with just a suffix array?



ABANANABANDANA$

Longest Repeated Substring

- We can list all branching words from a suffix array in time O($\boldsymbol{m}^2$).
  - O($\boldsymbol{m}$) pairs; each pair takes time O($\boldsymbol{m}$) to process.
- This worst-case bound can be realized.
- Contrast this with O($\boldsymbol{m}$) for a suffix tree.
- Can we do better?

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NDANA$

ABANANABANDANA$

Longest Repeated Substring

- ***Observation:*** We don't actually need to know what all the branching words are to find the longest repeated substring.
- We just need to know how long they are.
- That way, we can figure out which is longest.
- Is there some nice way to do this?

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A$
ABANANABANDANA$
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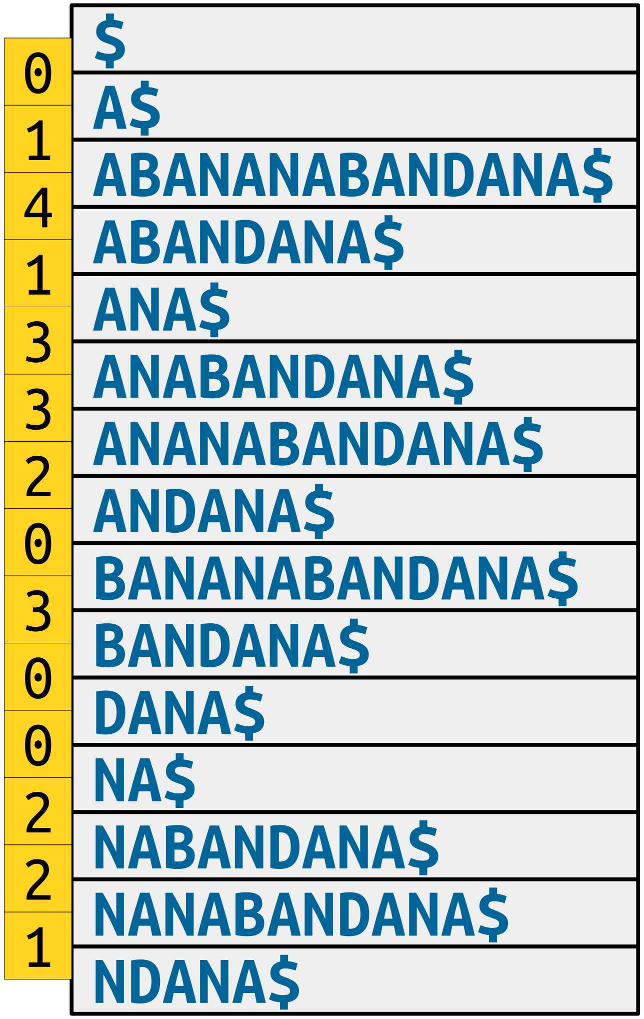
LCP Arrays

LCP Arrays

- The ***LCP array***, often denoted ***H***, is an array where $H[i]$ is the length of the LCP of the ith and $(i+1)$st suffixes in the suffix array.
- (The letter H comes from “height.”)



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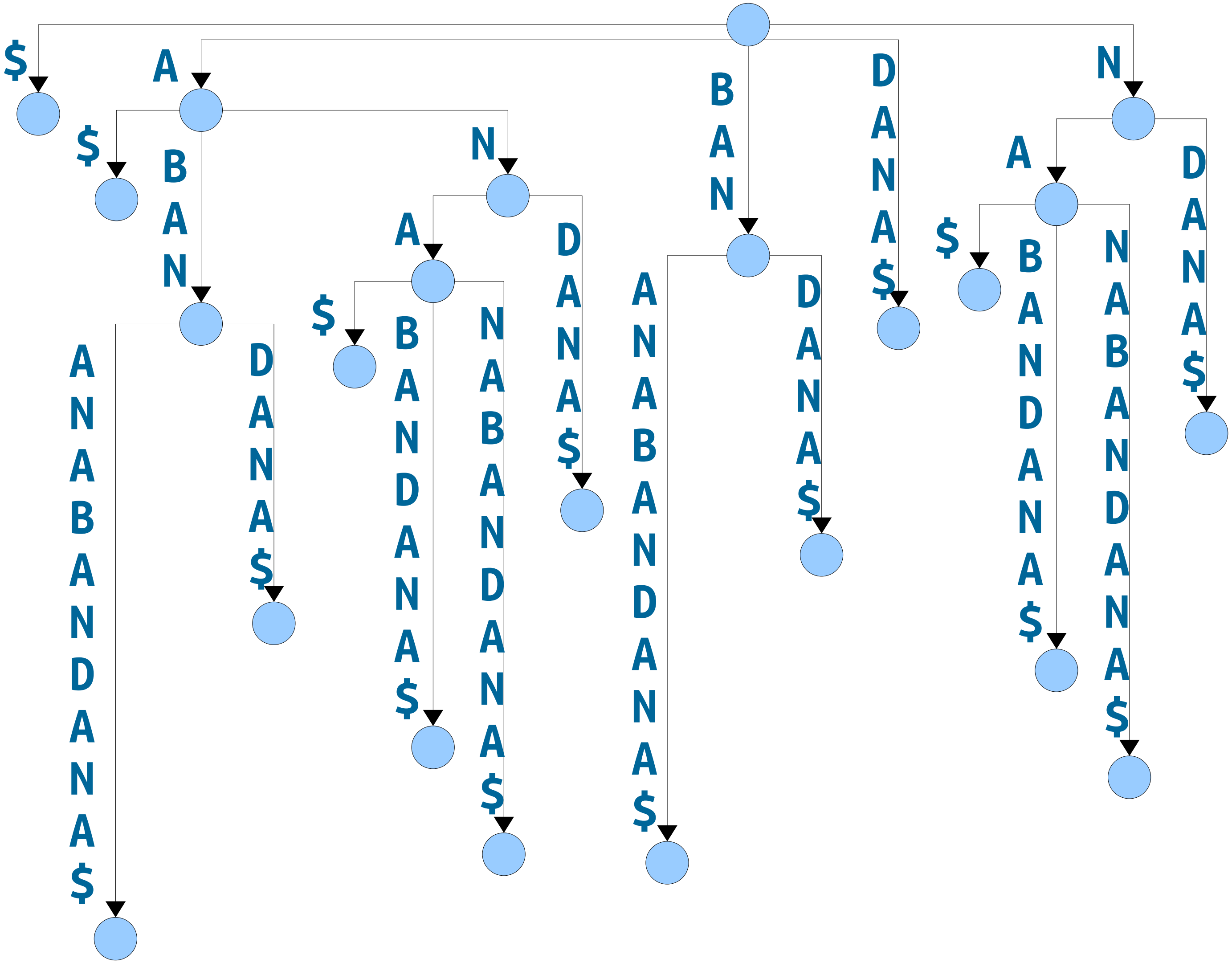
0	A$
1	ABANANABANDANA$
4	ABANDANA$
1	ANA$
3	ANABANDANA$
3	ANANABANDANA$
2	ANDANA$
0	BANANABANDANA$
3	BANDANA$
0	DANA$
0	NA$
2	NABANDANA$
2	NANABANDANA$
1	NDANA$

ABANANABANDANA$



	$
0	A$
1	ABANANABANDANA$
4	ABANDANA$
1	ANA$
3	ANABANDANA$
3	ANANABANDANA$
2	ANDANA$
0	BANANABANDANA$
3	BANDANA$
0	DANA$
0	NA$
2	NABANDANA$
2	NANABANDANA$
1	NDANA$

ABANANABANDANA$

Key intuition: The suffix array gives the leaves of the suffix tree.
The LCP array gives the internal nodes of the suffix tree.

Using LCP Arrays

- If you already have a suffix array and LCP array, you can solve longest repeated substring in time O(***m***):
  - Find the largest element in the LCP array.
  - Return the string it corresponds to.
- ***Question:*** How fast can we construct an LCP array?

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0	A$
1	ABANANABANDANA$
4	ABANDANA$
1	ANA$
3	ANABANDANA$
3	ANANABANDANA$
2	ANDANA$
0	BANANABANDANA$
3	BANDANA$
0	DANA$
0	NA$
2	NABANDANA$
2	NANABANDANA$
1	NDANA$

ABANANABANDANA$

Building LCP Arrays

Building LCP Arrays

- It never hurts to start with the naive algorithm and see what happens!
- ***Algorithm:*** For each consecutive pair of strings in the suffix array, compute the length of their longest common prefix.
- We can upper-bound the runtime at O(***m***^{2}).
- ***Question:*** Can we realize this upper bound?

	$
0	A$
1	ABANANABANDANA$
4	ABANDANA$
1	ANA$
3	ANABANDANA$
3	ANANABANDANA$
2	ANDANA$
0	BANANABANDANA$
3	BANDANA$
0	DANA$
0	NA$
2	NABANDANA$
2	NANABANDANA$
1	NDANA$

ABANANABANDANA$

Building LCP Arrays

- Why is our naive algorithm slow?
- ***Intuition:*** We aren’t able to carry work from one suffix over to the next.

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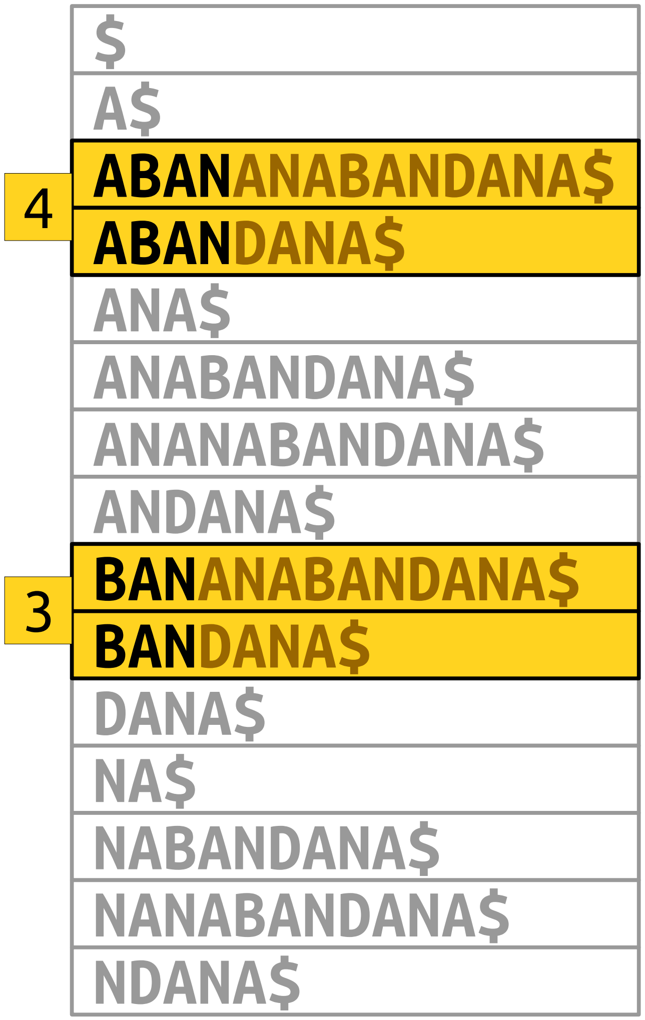
0	A$
1	ABANANABANDANA$
4	ABANDANA$
1	ANA$
3	ANABANDANA$
3	ANANABANDANA$
2	ANDANA$
0	BANANABANDANA$
3	BANDANA$
0	DANA$
0	NA$
2	NABANDANA$
2	NANABANDANA$
1	NDANA$

ABANANABANDANA$

Building LCP Arrays

- ***Key intuition:*** Suffixes overlap one another! It should be possible to share LCP information across suffixes.
- For example, suppose we compute the LCP entry shown here.
- Look at the suffixes formed by dropping the first letter of these two suffixes.
- What do we know about their LCP?



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ABANANABANDANA$

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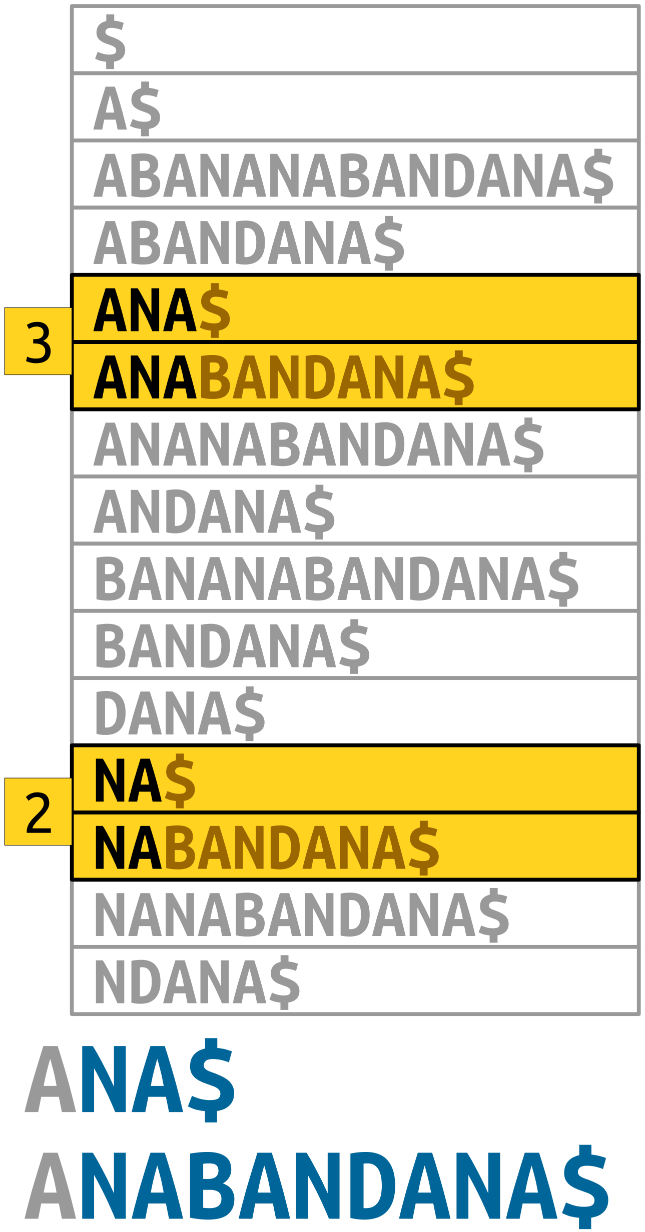
Building LCP Arrays

- Let’s do another example. Suppose we know the LCP of these suffixes.
- As before, drop the first letter from each suffix.
- What can we say about the LCP of the resulting suffixes?



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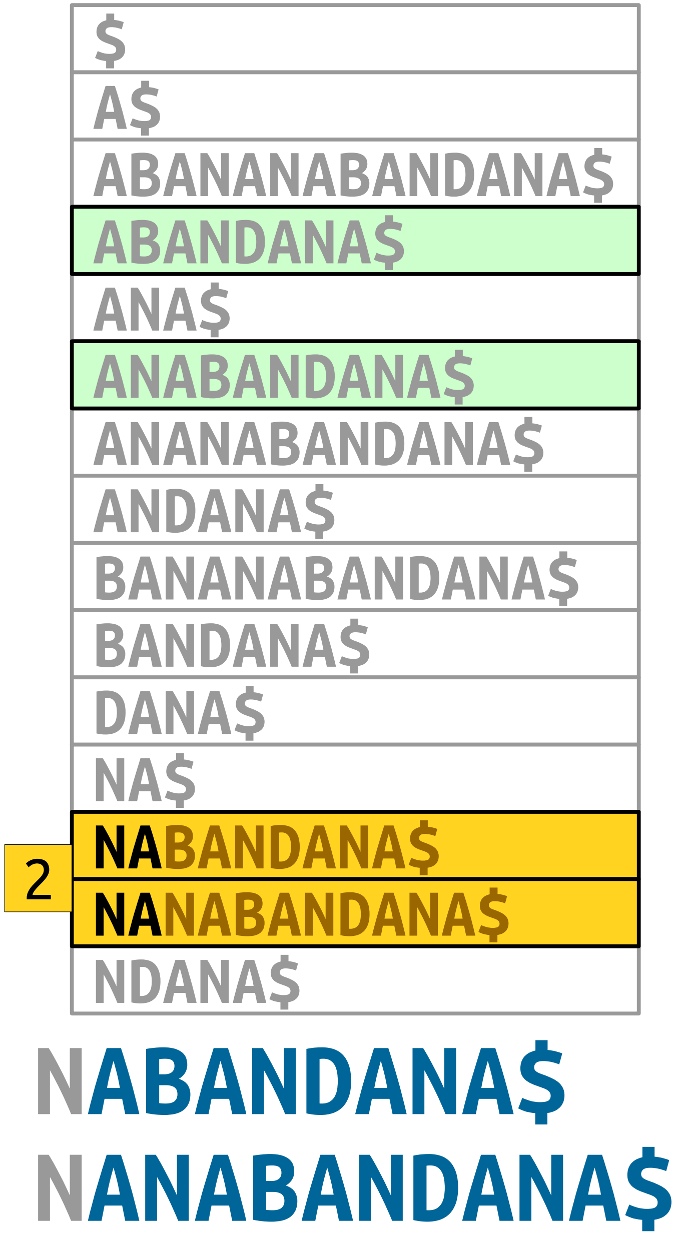
Building LCP Arrays

- Sometimes, in dropping the first letter, two adjacent suffixes get spread out.



$

A$

ABANANABANDANA$

ABANDANA$

ANA$

ANABANDANA$

ANANABANDANA$

ANDANA$

BANANABANDANA$

BANDANA$

DANA$

NA$

NABANDANA$

2

NANABANDANA$

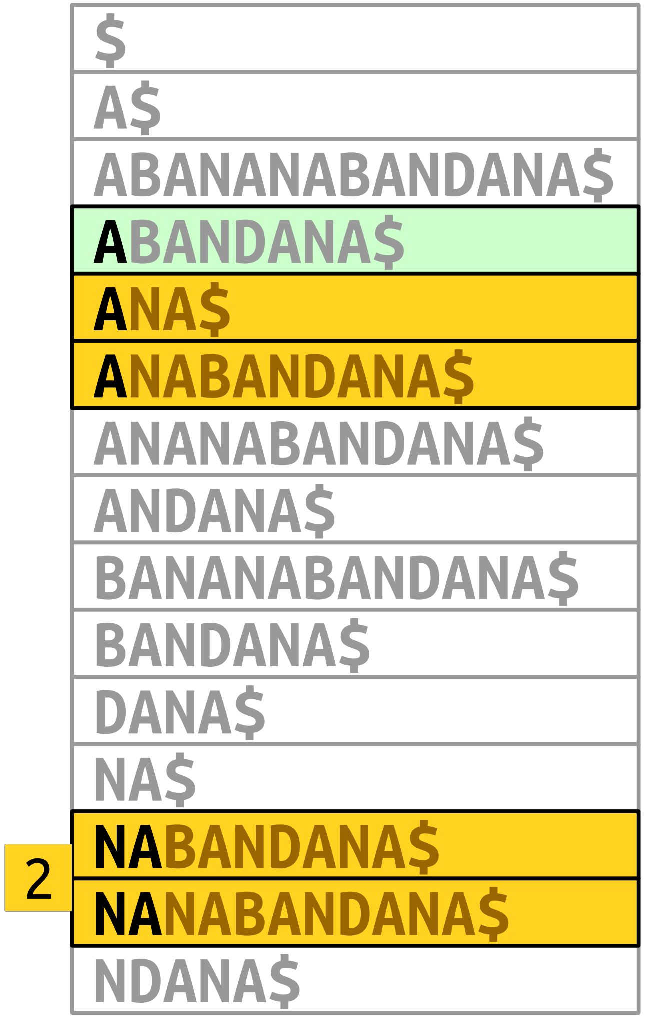
NDANA$

NABANDANA$

NANABANDANA$

Building LCP Arrays

- Sometimes, in dropping the first letter, two adjacent suffixes get spread out.
- ***Claim:*** Look at the second suffix in the pair. Its LCP with the suffix before it is at least the previous LCP minus one.
- Think about the suffix tree. The two shorter suffixes are in the same subtree, so everything between them is also in that subtree.



$
A$
ABANANABANDANA$
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ANABANDANA$
ANANABANDANA$
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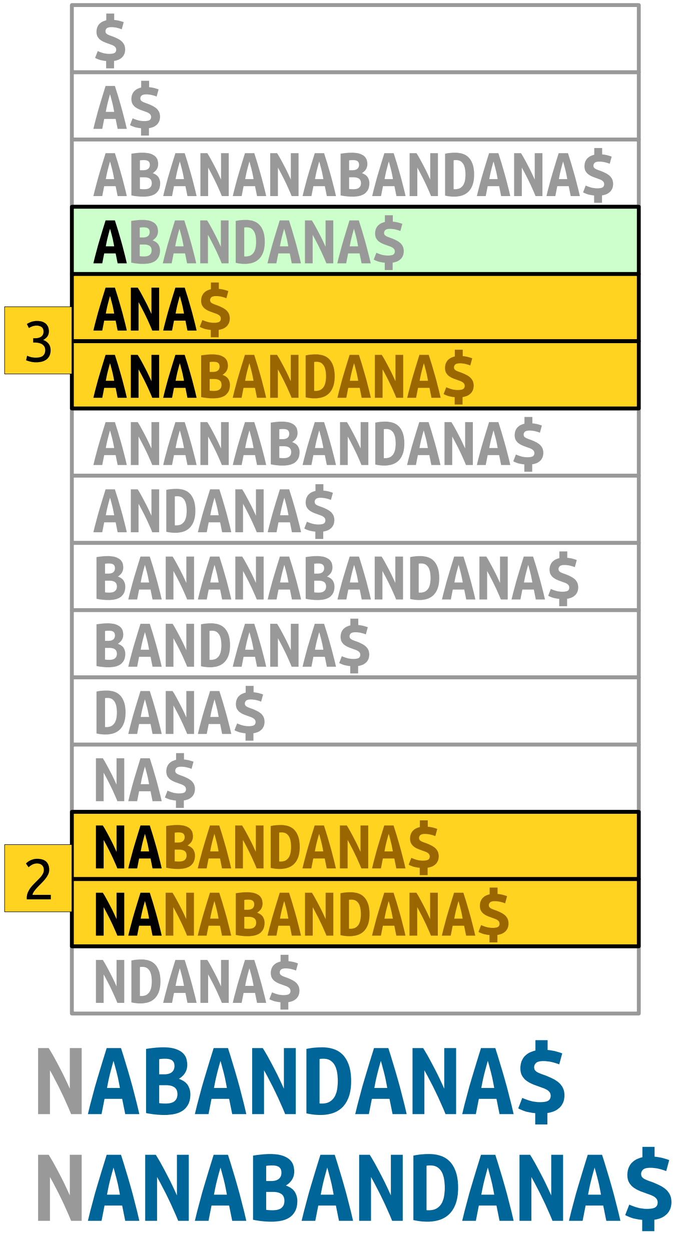
BANDANA$
DANA$
NA$
2 NABANDANA$
NANABANDANA$
NDANA$

NABANDANA$
NANABANDANA$

Building LCP Arrays

- We know that these two new suffixes must have an LCP of at least 1, because the two old suffixes have an LCP of 2.
- However, the LCP may be longer than 1, since we've never seen one of these two suffixes.
- We still need to some some scanning, but we won't necessarily have to rescan the entire suffix.



$
A$
ABANANABANDANA$
ABANDANA$
3
ANA$
ANABANDANA$
ANANABANDANA$
ANDANA$
BANANABANDANA$
BANDANA$
DANA$
NA$
2
NABANDANA$
NANABANDANA$
NDANA$

NABANDANA$
NANABANDANA$

Kasai’s Algorithm

- For each suffix of the original string, except the last:
  - Find that suffix in the suffix array.
  - Look at the suffix that comes before it.
  - (★) Find the length of the longest common prefix of those suffixes.
  - Write that down in the *H* array.
- Use the insight from the previous slides to speed up step (★).

$
A$
ABANANABANDANA$
ABANDANA$
ANA$
ANABANDANA$
ANANABANDANA$
ANDANA$
BANANABANDANA$
BANDANA$
DANA$
NA$
NABANDANA$
NANABANDANA$
NDANA$

ABANANABANDANA$

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	$
0	A$
1	ABANANABANDANA$
4	ABANDANA$
1	ANA$
3	ANABANDANA$
3	ANANABANDANA$
2	ANDANA$
0	BANANABANDANA$
3	BANDANA$
0	DANA$
0	NA$
2	NABANDANA$
2	NANABANDANA$
1	NDANA$

ABANANABANDANA$

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With O(m) preprocessing time, can be done in time O(1).

Question to Ponder:
How would you do this?

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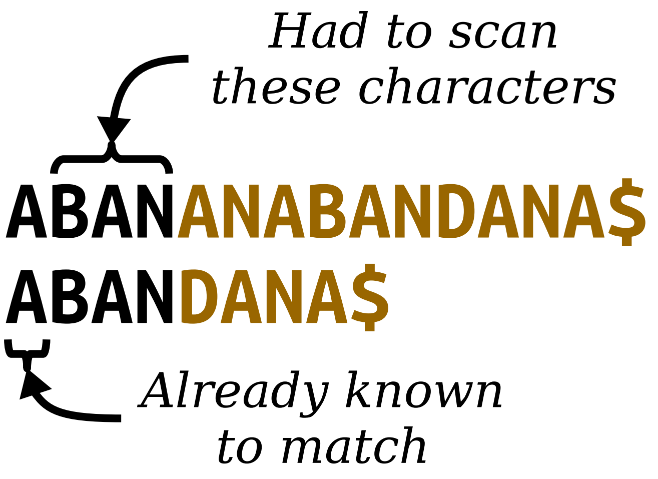
The runtime of this step is proportional to how much the LCP increases on that step.



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The runtime of this step is proportional to how much the LCP increases on that step.

The LCP value decreases by at most one per suffix. *(We saw this earlier.)*

The LCP value maxes out at m. *(Can't match more than all the characters.)*

Therefore, the LCP value can grow at most $2m$ times. *(Prove this!)*

Claim: Across all iterations, this step takes a total of O(m) time.

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Total runtime: O(***m***).

More to Explore

- We could easily spend a whole quarter talking about suffix arrays. Here’s what we didn’t cover:
  - ***Bottom-up tree simulations:*** Using LCP arrays, you can simulate any O($\boldsymbol{m}$)-time suffix tree algorithm that works with a bottom-up DFS in time O($\boldsymbol{m}$).
  - ***Faster substring searching:*** Using LCP arrays, plus RMQ, you can improve the cost of a substring search to O($\boldsymbol{n}$ + $\boldsymbol{z}$ + log $\boldsymbol{m}$).
  - ***Burrows-Wheeler transforms:*** Suffix arrays, plus LCP arrays, can be used to significantly improve the performance of text compressors.
- Check these out – they’re super interesting!

Next Time

- ***Amortized Analysis***
  - Lying in a runtime analysis.
- ***The Potential Method***
  - Physics meets data structure design.