CS166: Advanced Data Structures

Welcome!

Why study advanced data structures?

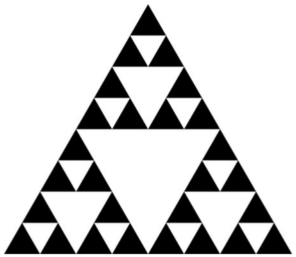
Why Study Advanced Data Structures?

- ***Expand your library of problem-solving tools.***
  - We'll cover a wide range of tools for a bunch of interesting problems. These come in handy, both IRL an in Theoryland.
- ***Learn new problem-solving techniques.***
  - We'll see some truly beautiful problem-solving strategies that work beyond just a single example.
- ***Challenge your intuition for the limits of efficiency.***
  - You'd be amazed how many times we'll take a problem you're sure you know how to solve and then see how to solve it faster.
- ***See the beauty of theoretical computer science.***
  - We'll cover some amazingly clever theoretical techniques in the course of this class. You'll love them.

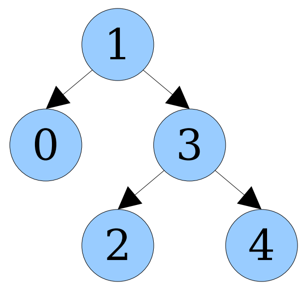
Where is CS166 situated in Stanford's CS sequence?

Our (Transitive) Prerequisites

CS106B / CS107



```
struct Node {
    int value;
    Node* left;
    Node* right;
};
```




```
make && gdb ./a.out
```

CS103

$a_0 = 1 \qquad a_{n+1} = 2a_n + n$

Theorem: $a_n = 2^{n+1} - n - 1$.

Proof: By induction. As a base case, when $n = 0$, we have

$$2^{n+1} - n - 1 = 2^1 - 0 - 1 = 1 = a_0.$$

For the inductive step, assume that $a_k = 2^{k+1} - k - 1$. Then

$$\begin{aligned} a_{k+1} &= 2a_k + k \\ &= 2^{k+2} - 2k - 2 + k \\ &= 2^{(k+1)+1} - (k+1) - 1, \end{aligned}$$

as required. ■

CS109

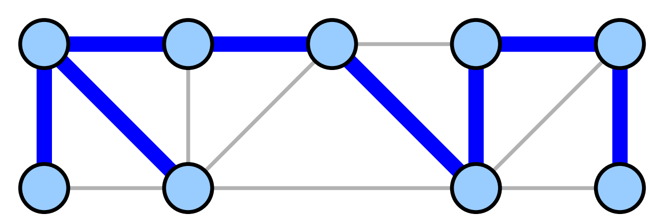
$$\mathrm{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathrm{E}[X_i]$$

$$\Pr[X \geq c] \leq \frac{\mathrm{E}[X]}{c}$$

CS161

$$\mathrm{T}(n) = a\mathrm{T}(n / b) + \mathrm{O}(n^d)$$

$$\begin{aligned} n^2 \log n^2 &= \mathrm{O}(n^3) \\ n^2 \log n^2 &= \Omega(n^2) \\ n^2 \log n^2 &= \Theta(n^2 \log n) \end{aligned}$$



Who are we?

Course Staff

Keith Schwarz (htiek@cs.stanford.edu)

Kevin Tan

Ping us over EdStem with questions!

The Course Website

https://cs166.stanford.edu

Course Requirements

- We plan on having six ***problem sets***.
  - Problem sets may be completed individually or in a pair. (Exception: PS0 must be done individually.)
  - They're a mix of written problems and C++ coding exercises.
  - You'll submit one copy of the problem set regardless of how many people worked on it.
  - Need to find a partner? Use EdStem, stop by office hours, or send us an email.
- We plan on having a ***midterm exam***.
  - The plan is to hold it on Tuesday, May 30th from 7:00PM – 10:00PM.
- We plan on requiring ***lecture participation***.
  - This will help build community and improve learning outcomes.
  - We'll use PollEV for in-class questions starting in Week 3.
- Why "plan on?" Two reasons.

Problem Set 0

- Problem Set 0 goes out today. It’s due next Tuesday at noon Pacific time.
- This is mostly designed as a refresher of topics from the prerequisite courses CS103, CS107, CS109, and CS161.
- If you’re mostly comfortable with these problems and are just “working through some rust,” then you’re probably in the right place!

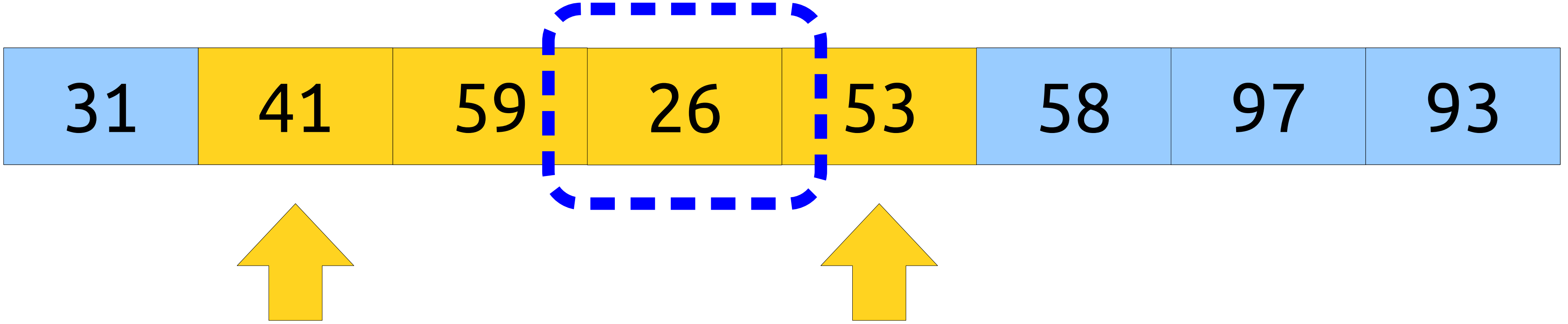
Let’s Get Started!

Range Minimum Queries

The RMQ Problem

- The ***Range Minimum Query problem*** (***RMQ*** for short) is the following:

  Given an array A and two indices $i \leq j$, what is the smallest element out of A[i], A[i + 1], …, A[j – 1], A[j]?



The RMQ Problem

- The ***Range Minimum Query problem*** (***RMQ*** for short) is the following:

  Given an array A and two indices $i \leq j$, what is the smallest element out of $A[i], A[i+1], \ldots, A[j-1], A[j]$?

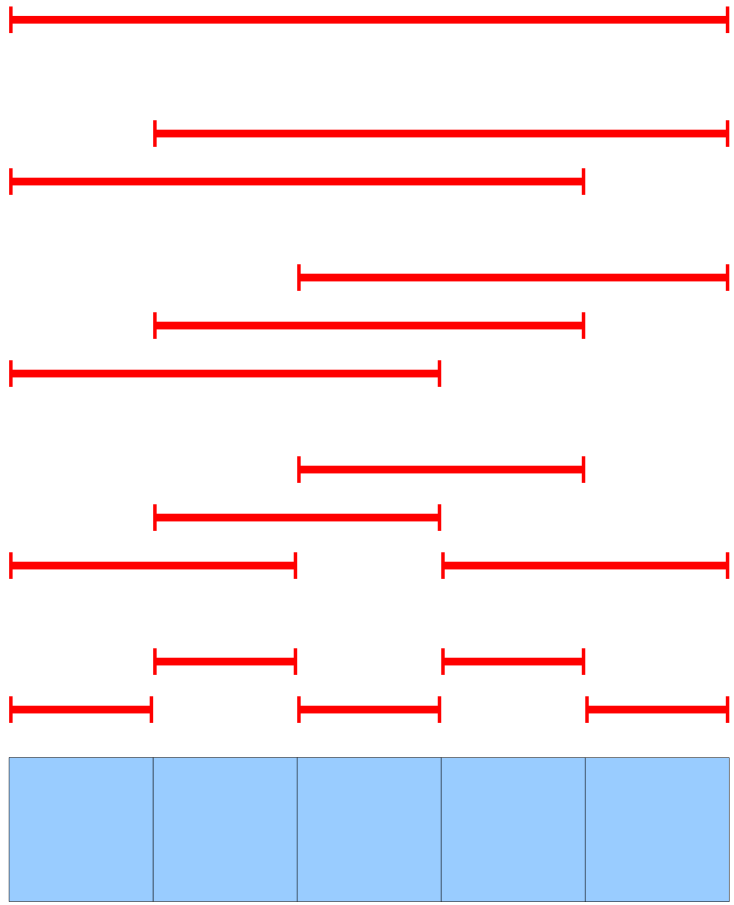
- Notation: We'll denote a range minimum query in array A between indices i and j as $\mathbf{RMQ_A}(i, j)$.
- For simplicity, let's assume 0-indexing.

A Trivial Solution

- There's a simple O(n)-time algorithm for evaluating $\text{RMQ}_A(i, j)$: just iterate across the elements between i and j, inclusive, and take the minimum!
- So… why is this problem at all algorithmically interesting?
- Suppose that the array A is fixed in advance and you're told that we're going to make multiple queries on it.
- Can we do better than the naïve algorithm?

An Observation

- In an array of length n, there are only $\Theta(n^2)$ distinct possible queries.
- Why?



1 *subarray of length 5*

2 *subarrays of length 4*

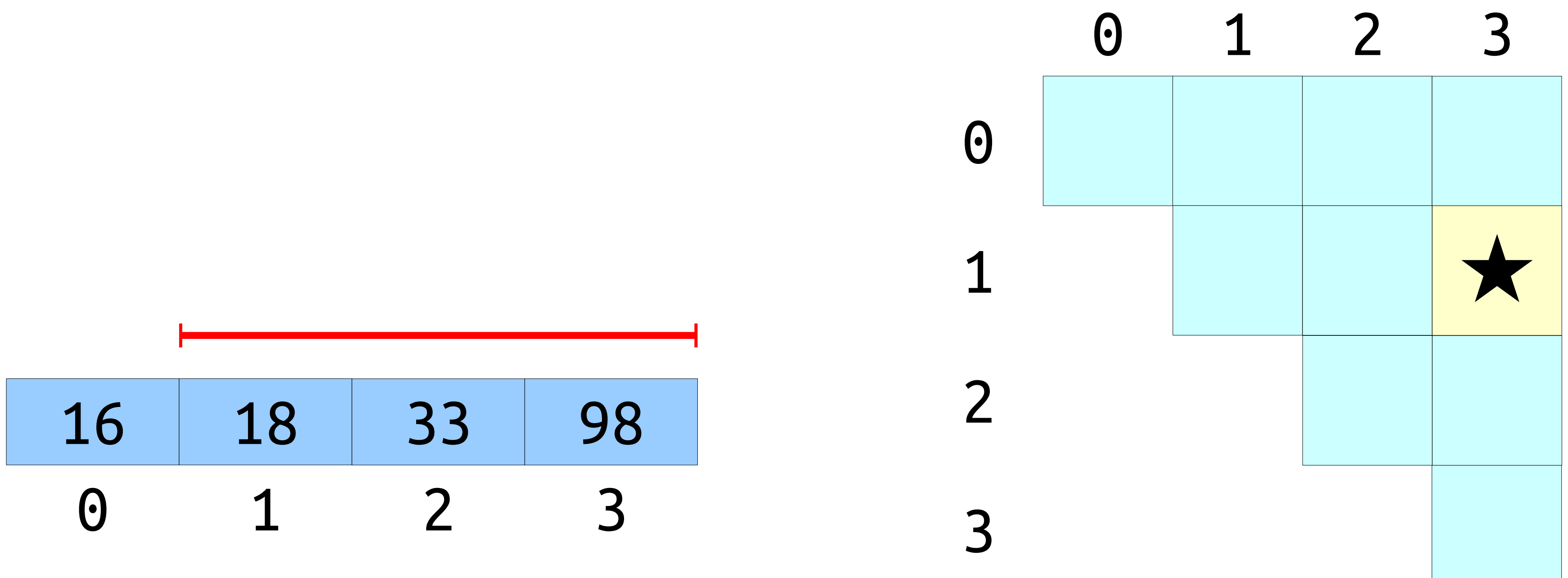
3 *subarrays of length 3*

4 *subarrays of length 2*

5 *subarrays of length 1*

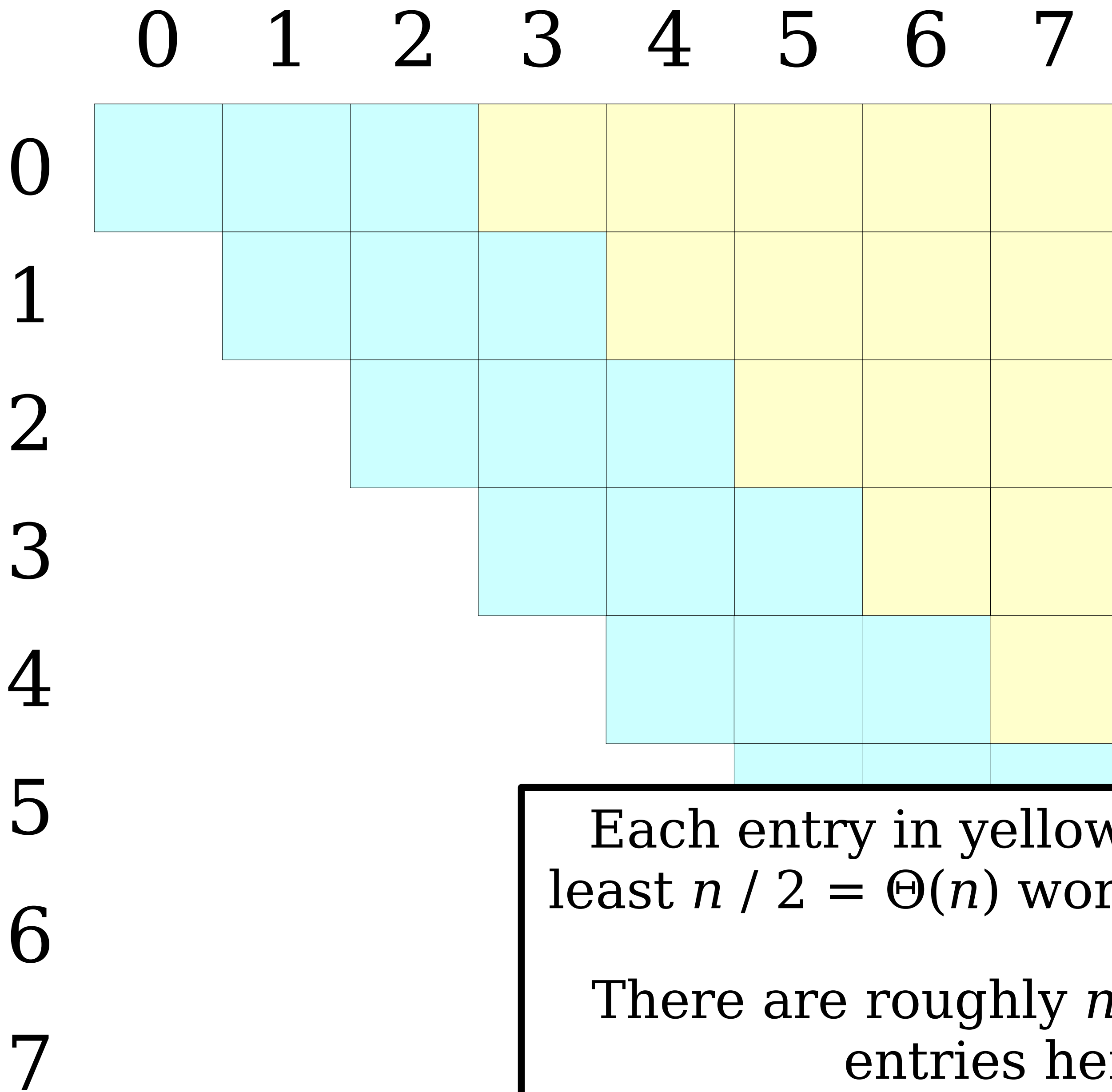
A Different Approach

- There are only $\Theta(n^2)$ possible RMQs in an array of length n.
- If we precompute all of them, we can answer RMQ in time O(1) per query.



Building the Table

- One simple approach: for each entry in the table, iterate over the range in question and find the minimum value.
- How efficient is this?
  - Number of entries: $\Theta(n^2)$.
  - Time to evaluate each entry: $O(n)$.
  - Time required: $O(n^3)$.
- The runtime is $O(n^3)$ using this approach. Is it also $\Theta(n^3)$?



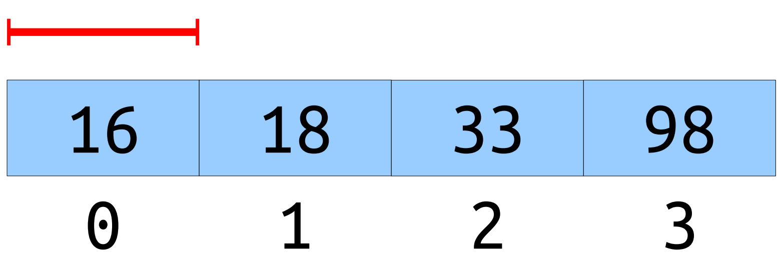
Each entry in yellow requires at least n / 2 = $\Theta(n)$ work to evaluate.

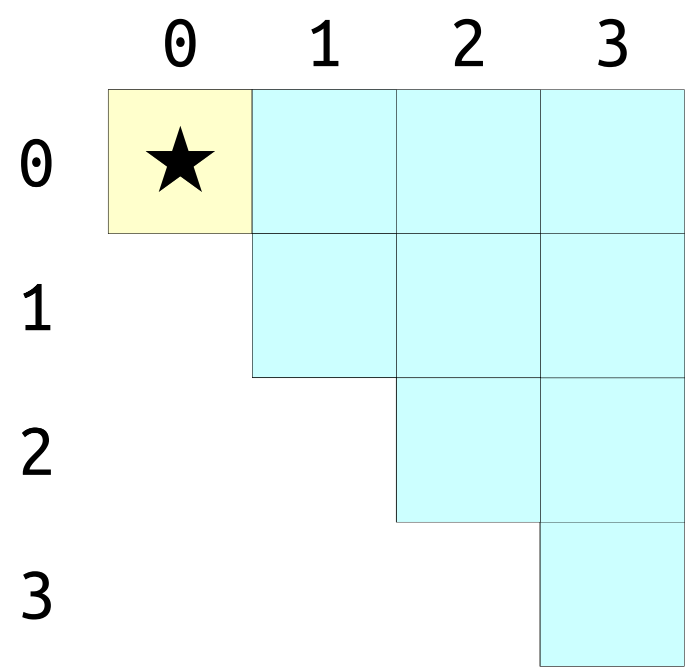
There are roughly n^2 / 8 = $\Theta(n^2)$ entries here.

Total work required: $\Theta(n^3)$

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.





A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

	0	1	2	3
0	16			
1				
2				
3				

16	18	33	98
0	1	2	3

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16			
1		★		
2				
3				

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16			
1		18		
2				
3				

A Different Approach

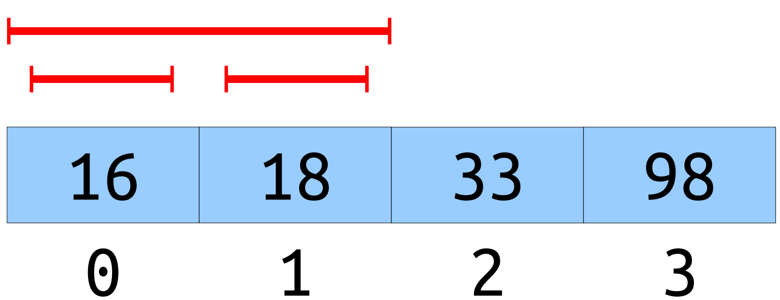
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

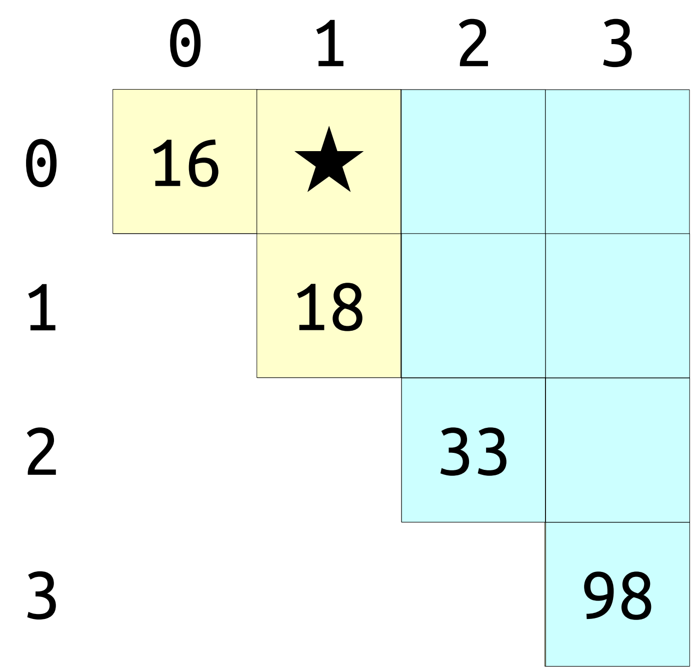
16	18	33	98
0	1	2	3

	0	1	2	3
0	16			
1		18		
2			33	
3				98

A Different Approach

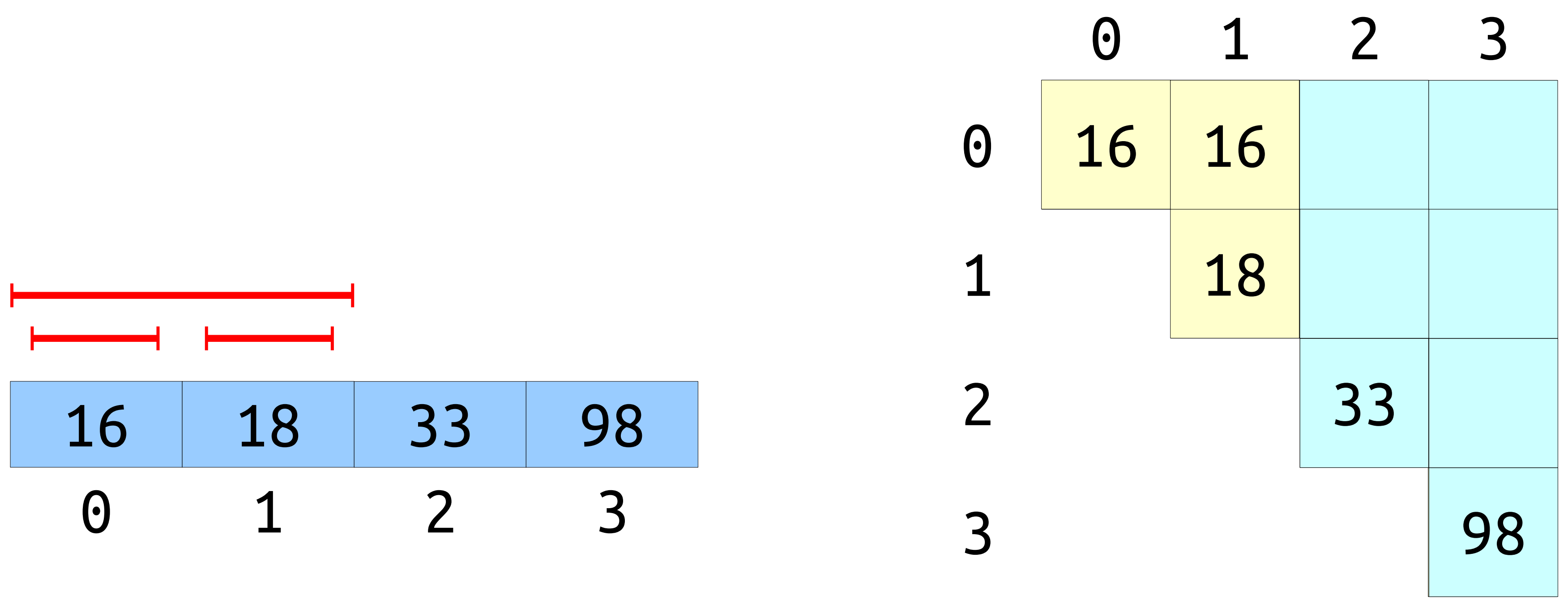
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.





A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



A Different Approach

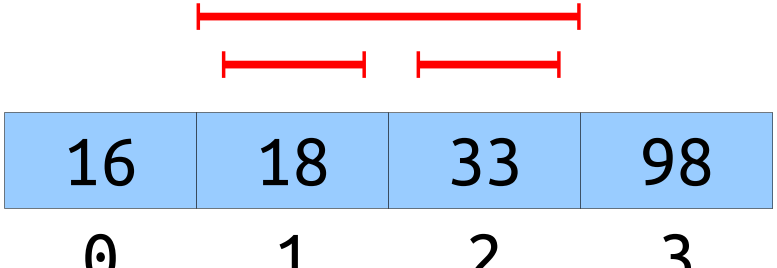
- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16	16		
1		18	★	
2			33	
3				98

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.



	0	1	2	3
0	16	16		
1		18	18	
2			33	
3				98

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

	0	1	2	3
0	16	16		
1		18	18	
2			33	33
3				98

16	18	33	98
0	1	2	3

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16	16	★	
1		18	18	
2			33	33
3				98

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16	16	16	16
1		18	18	18
2			33	33
3				98

Some Notation

- We'll say that an RMQ data structure has time complexity **$\langle p(n), q(n) \rangle$** if
  - preprocessing takes time at most $p(n)$ and
  - queries take time at most $q(n)$.
- We now have two RMQ data structures:
  - $\langle O(1), O(n) \rangle$ with no preprocessing.
  - $\langle O(n^2), O(1) \rangle$ with full preprocessing.
- These are two extremes on a curve of tradeoffs: no preprocessing versus full preprocessing.
- ***Question:*** *Is there a "golden mean" between these extremes?*

Another Approach: ***Block Decomposition***

A Block-Based Approach

- Split the input into O(n / b) blocks of some "block size" b.
  - Here, b = 4.

31	41	59	26	53	58	97	93	23	84	62	43	33	83	27	95	2	88	41	97

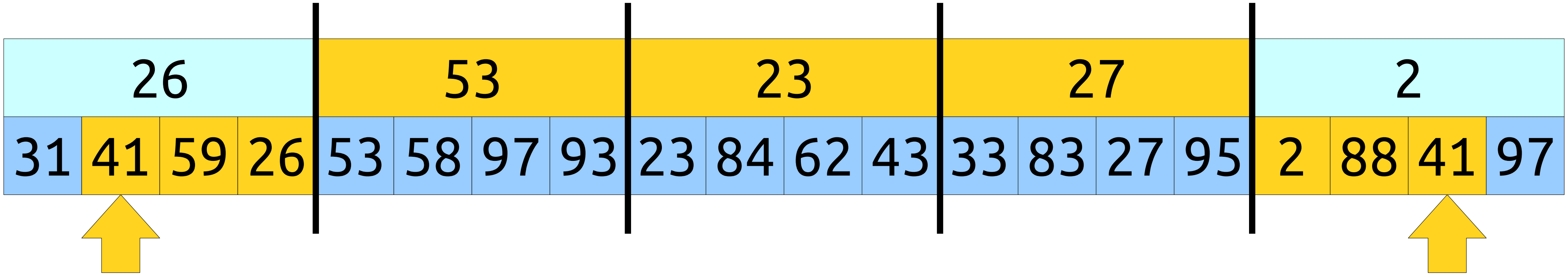
A Block-Based Approach

- Split the input into O(n / b) blocks of some “block size” b.
  - Here, b = 4.
- Compute the minimum value in each block.

26				53				23				27				2			
31	41	59	26	53	58	97	93	23	84	62	43	33	83	27	95	2	88	41	97

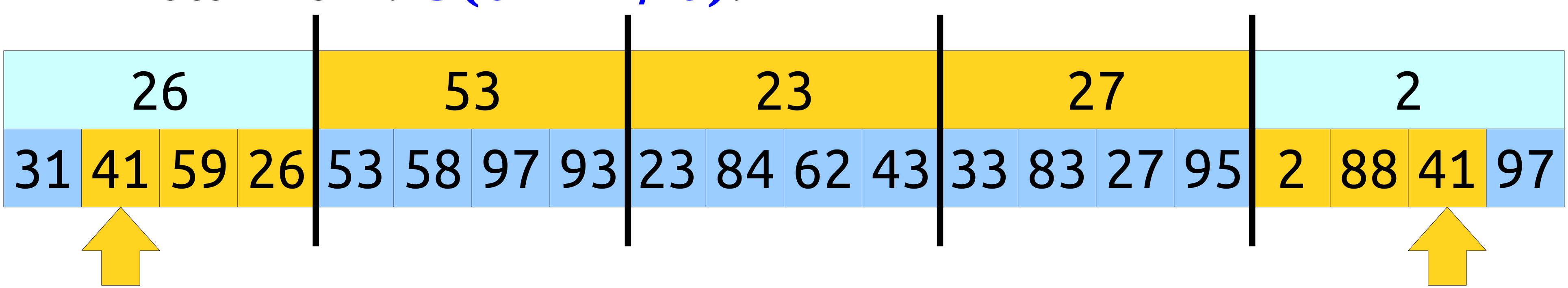
A Block-Based Approach

- Split the input into O(n / b) blocks of some “block size” b.
  - Here, b = 4.
- Compute the minimum value in each block.



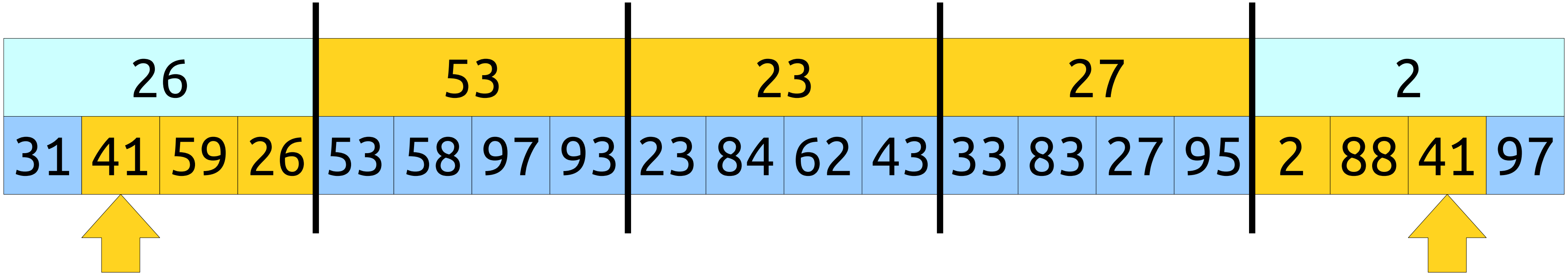
Analyzing the Approach

- Let's analyze this approach in terms of n and b.
- Preprocessing time:
  - O(b) work on O(n / b) blocks to find minima.
  - Total work: **O(n)**.
- Time to evaluate $\text{RMQ}_A(i, j)$:
  - O(1) work to find block indices (divide by block size).
  - O(b) work to scan inside i and j's blocks.
  - O(n / b) work looking at block minima between i and j.
  - Total work: **O(b + n / b)**.



Intuiting O(*b* + *n* / *b*)

- As *b* increases:
  - The ***b*** term rises (more elements to scan within each block).
  - The ***n* / *b*** term drops (fewer blocks to look at).
- As *b* decreases:
  - The ***b*** term drops (fewer elements to scan within a block).
  - The ***n* / *b*** term rises (more blocks to look at).
- Is there an optimal choice of *b* given these constraints?



Optimizing *b*

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) = 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

$$\begin{aligned} 1-n/b^2 &= 0 \\ 1 &= n/b^2 \\ b^2 &= n \\ b &= \sqrt{n} \end{aligned}$$

- Asymptotically optimal runtime is when $b = n^{1/2}$.
- In that case, the runtime is

  $O(b + n / b) = O(n^{1/2} + n / n^{1/2}) = O(n^{1/2} + n^{1/2}) = \mathbf{O(n^{1/2})}$

Summary of Approaches

- Three solutions so far:
  - Full preprocessing: $\langle O(n^2),\ O(1)\rangle$.
  - Block partition: $\langle O(n),\ O(n^{1/2})\rangle$.
  - No preprocessing: $\langle O(1),\ O(n)\rangle$.
- Modest preprocessing yields modest performance increases.
- ***Question:*** Can we do better?

A Second Approach: ***Sparse Tables***

An Intuition

- The $\langle O(n^2), O(1)\rangle$ solution gives fast queries because every range we might look up has already been precomputed.
- This solution is slow overall because we have to compute the minimum of every possible range.
- ***Question:*** Can we still get constant-time queries without preprocessing all possible ranges?

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26	26	26	26	26
1		41	41	26	26	26	26	26
2			59	26	26	26	26	26
3				26	26	26	26	26
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

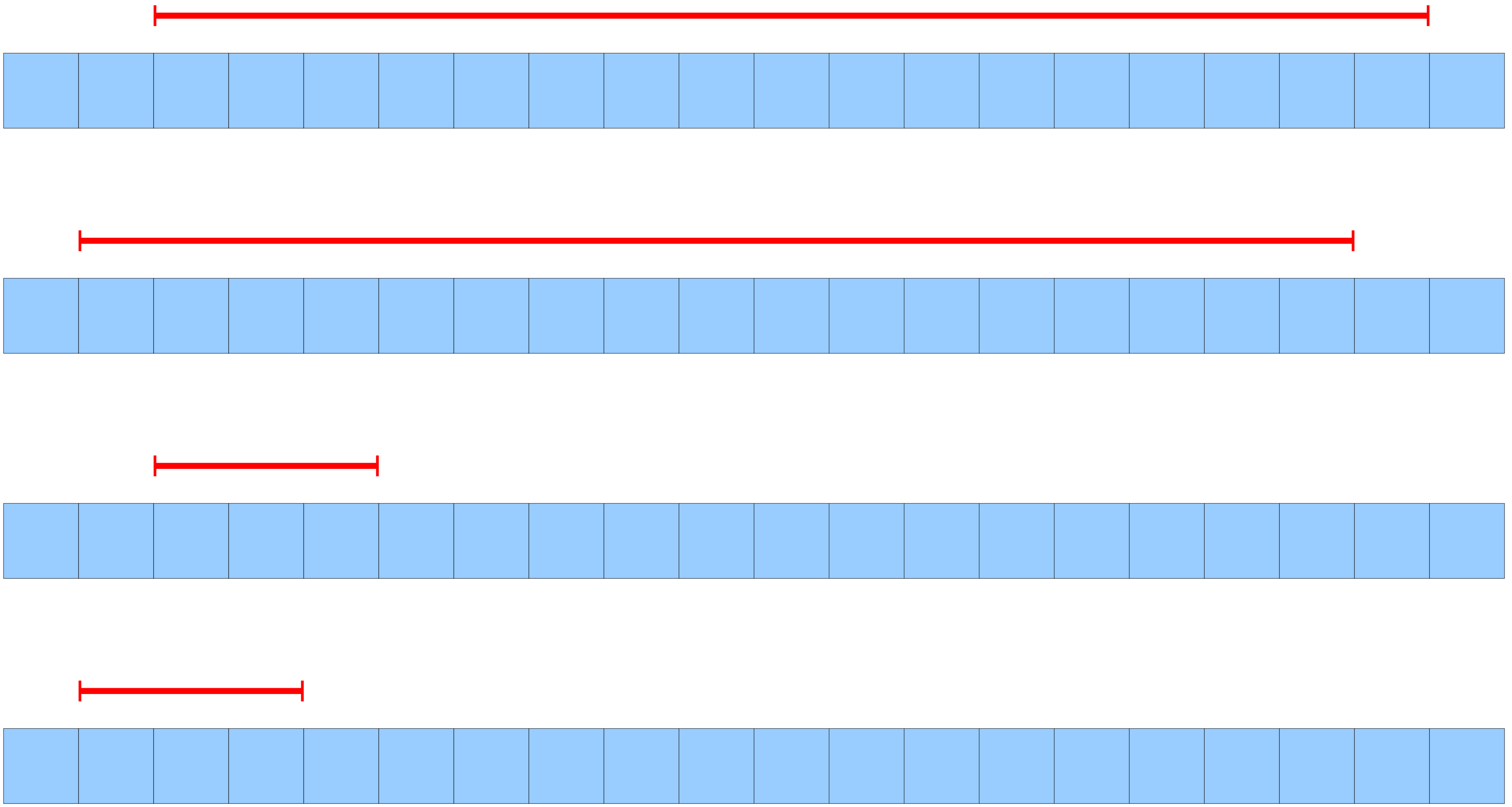
31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		★
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

The Intuition

- It's still possible to answer any query in time O(1) without precomputing RMQ over all ranges.
- If we precompute the answers over too many ranges, the preprocessing time will be too large.
- If we precompute the answers over too few ranges, the query time won't be O(1).
- ***Goal:*** Precompute RMQ over a set of ranges such that
  - there are $o(n^2)$ total ranges, but
  - there are enough ranges to support O(1) query times.

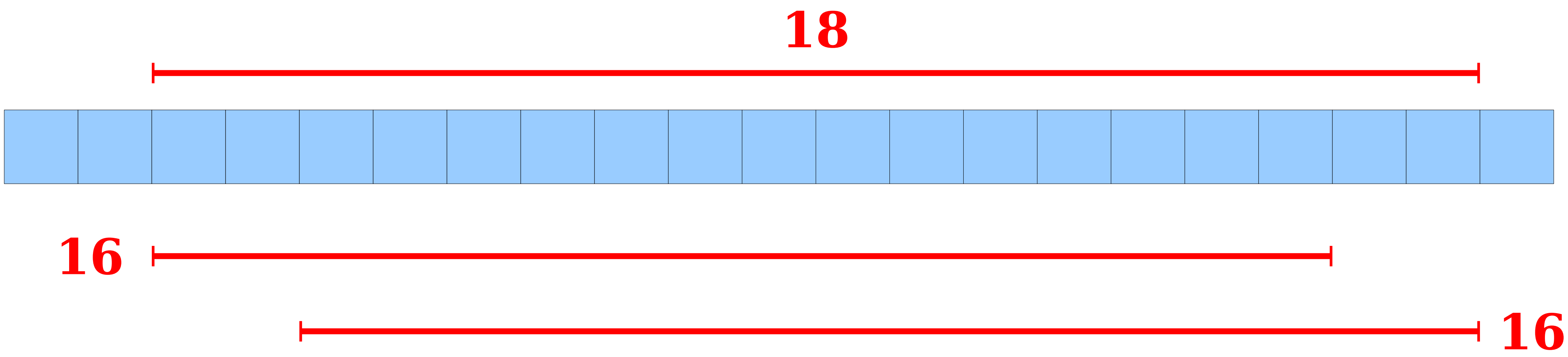
Some Observations



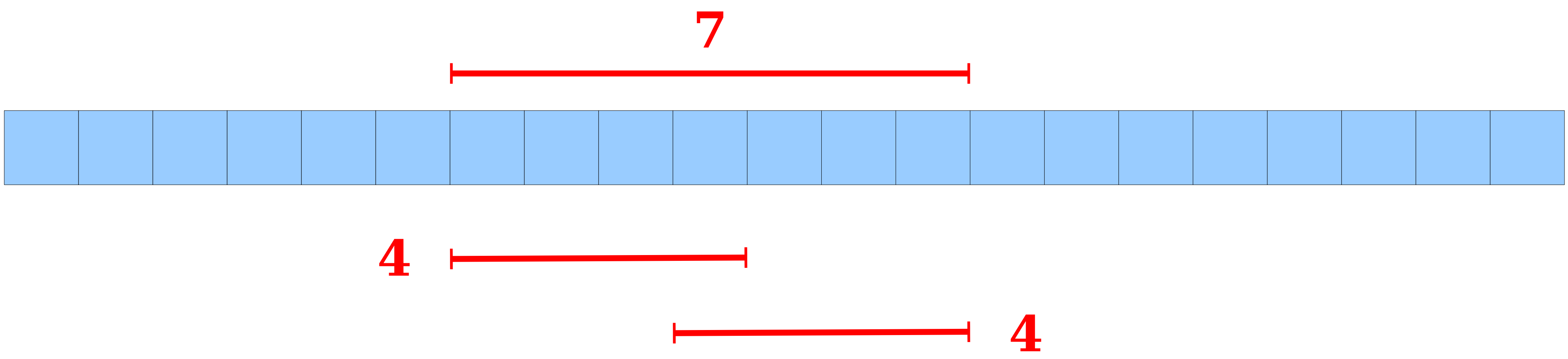
The Approach

- For each index *i*, compute RMQ for ranges starting at *i* of size 1, 2, 4, 8, 16, …, 2^k as long as they fit in the array.
  - Gives both large and small ranges starting at any point in the array.
  - Only O(log *n*) ranges computed for each array element.
  - Total number of ranges: O(*n* log *n*).
- ***Claim:*** Any range in the array can be formed as the union of two of these ranges.

Creating Ranges



Creating Ranges

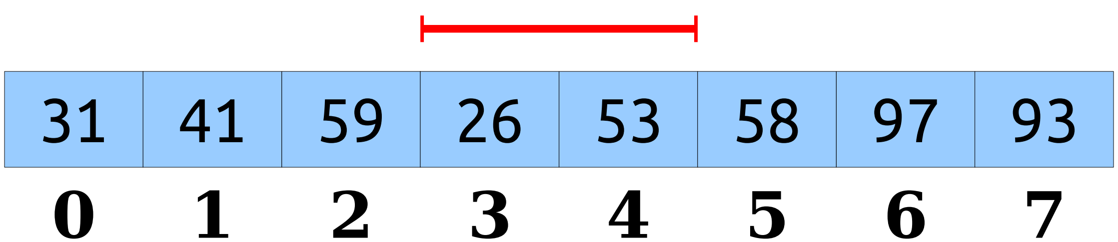


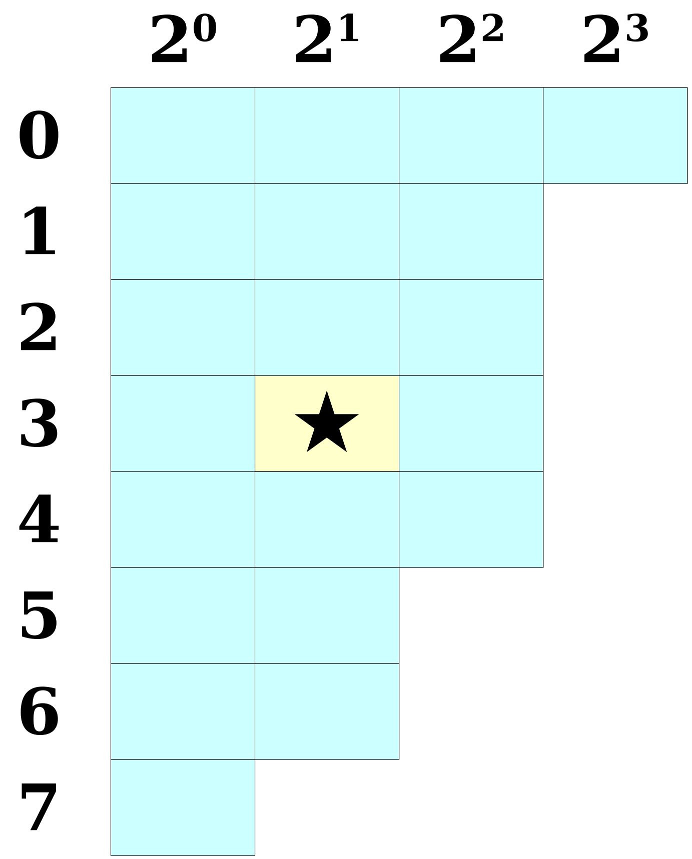
Doing a Query

- To answer $RMQ_A(i, j)$:
  - Find the largest k such that $2^k \leq j - i + 1$.
    - With the right preprocessing, this can be done in time O(1); you'll figure out how in an upcoming assignment.
  - The range $[i, j]$ can be formed as the overlap of the ranges $[i, i + 2^k - 1]$ and $[j - 2^k + 1, j]$.
  - Each range can be looked up in time O(1).
  - Total time: **O(1)**.

Precomputing the Ranges

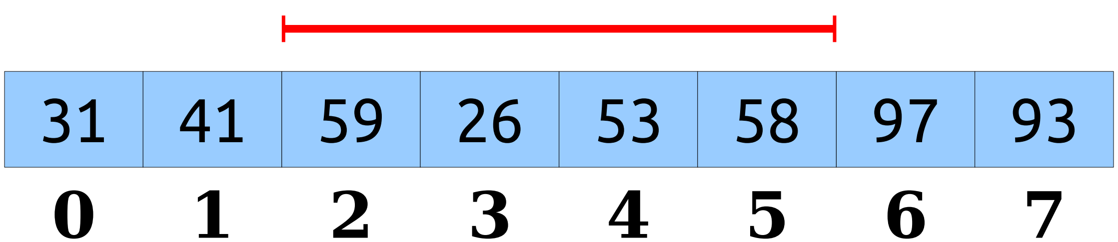
- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

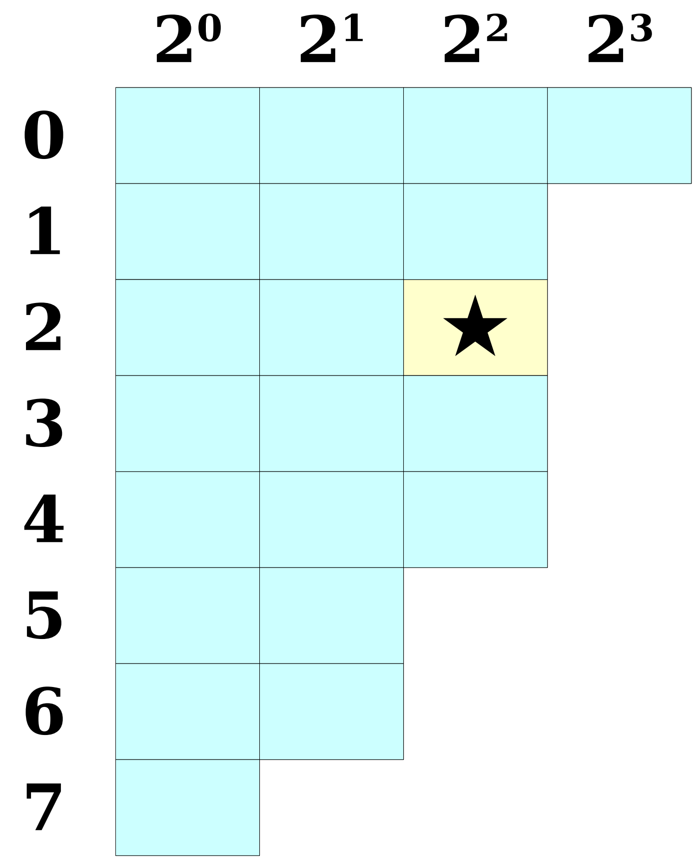




Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).





Precomputing the Ranges

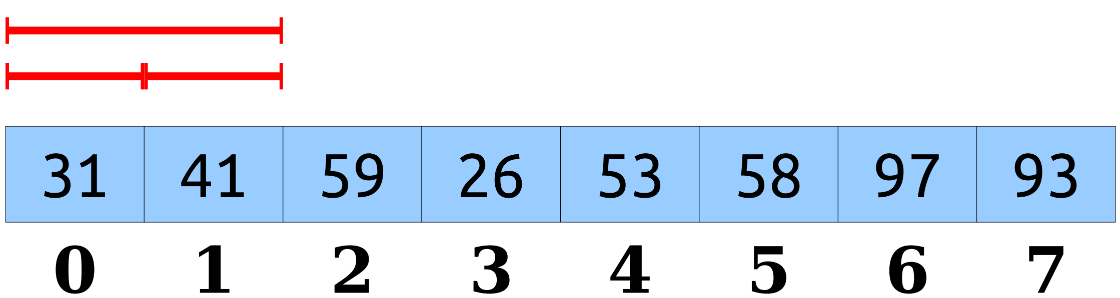
- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

	2^0	2^1	2^2	2^3
0	31			
1	41			
2	59			
3	26			
4	53			
5	58			
6	97			
7	93			

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).



	2^0	2^1	2^2	2^3
0	31	★		
1	41			
2	59			
3	26			
4	53			
5	58			
6	97			
7	93			

Precomputing the Ranges

- There are O(*n* log *n*) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(*n* log *n*).

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

	2^0	2^1	2^2	2^3
0	31	31		
1	41			
2	59			
3	26			
4	53			
5	58			
6	97			
7	93			

Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

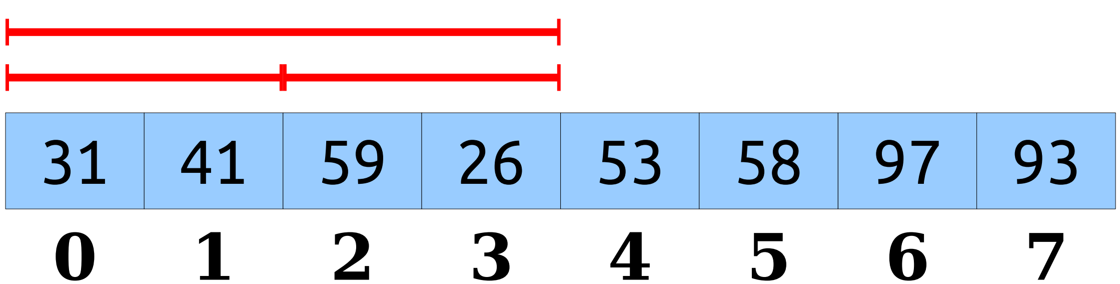
31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

	2^0	2^1	2^2	2^3
0	31	31		
1	41	41		
2	59	26		
3	26	26		
4	53	53		
5	58	58		
6	97	93		
7	93			

Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).

	2^0	2^1	2^2	2^3
0	31	31	★	
1	41	41		
2	59	26		
3	26	26		
4	53	53		
5	58	58		
6	97	93		
7	93			



Precomputing the Ranges

- There are $O(n \log n)$ ranges to precompute.
- Using dynamic programming, we can compute all of them in time $O(n \log n)$.

	2^0	2^1	2^2	2^3
0	31	31	26	26
1	41	41	26	
2	59	26	26	
3	26	26	26	
4	53	53	53	
5	58	58		
6	97	93		
7	93			

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Sparse Tables

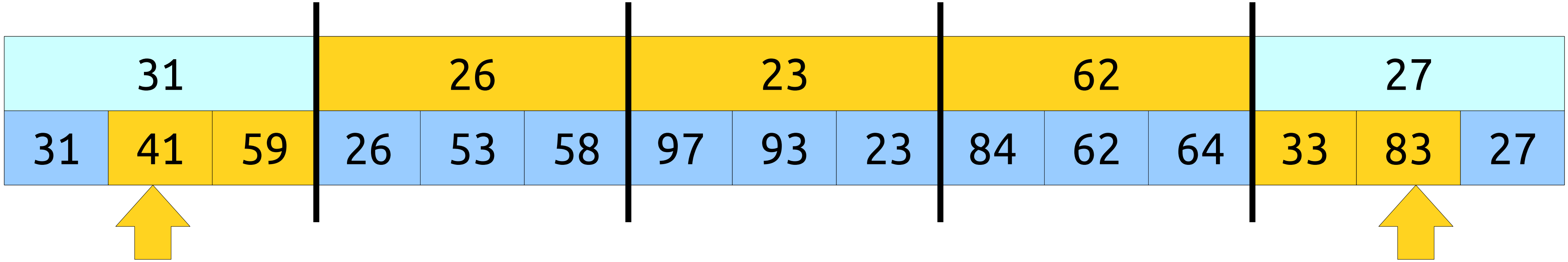
- This data structure is called a ***sparse table***.
- It gives an $\langle \mathbf{O}(\boldsymbol{n} \textbf{ log } \boldsymbol{n}), \mathbf{O}(1) \rangle$ solution to RMQ.
- This is asymptotically better than precomputing all possible ranges!

The Story So Far

- We now have the following solutions for RMQ:
  - Precompute all: $\langle O(n^2), O(1) \rangle$.
  - Sparse table: $\langle O(n \log n), O(1) \rangle$.
  - Blocking: $\langle O(n), O(n^{1/2}) \rangle$.
  - Precompute none: $\langle O(1), O(n) \rangle$.
- ***Can we do better?***

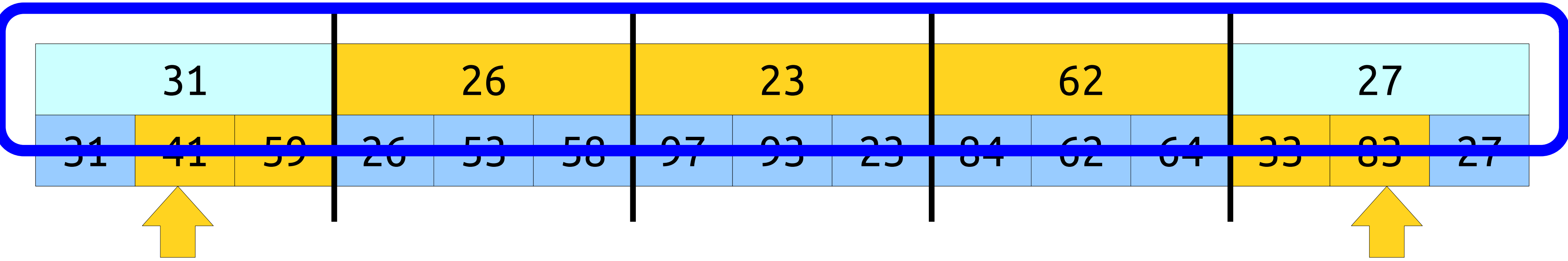
A Third Approach: ***Hybrid Strategies***

Blocking Revisited

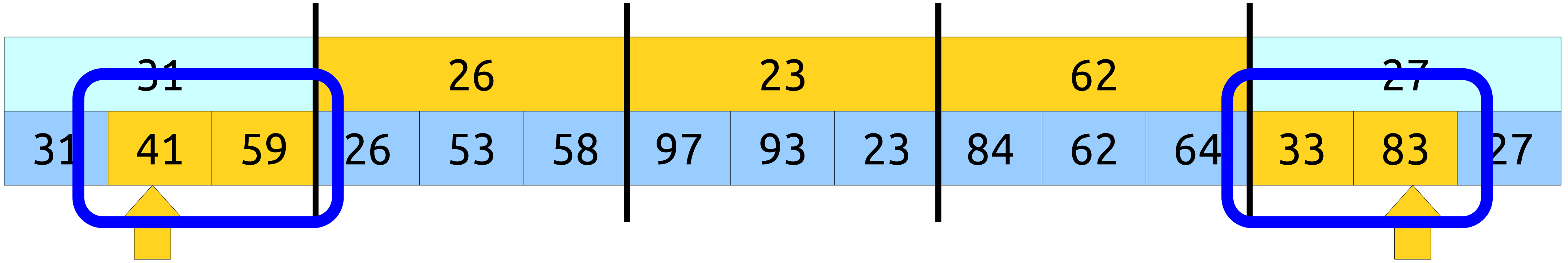


Blocking Revisited

This is just RMQ on the block minima!



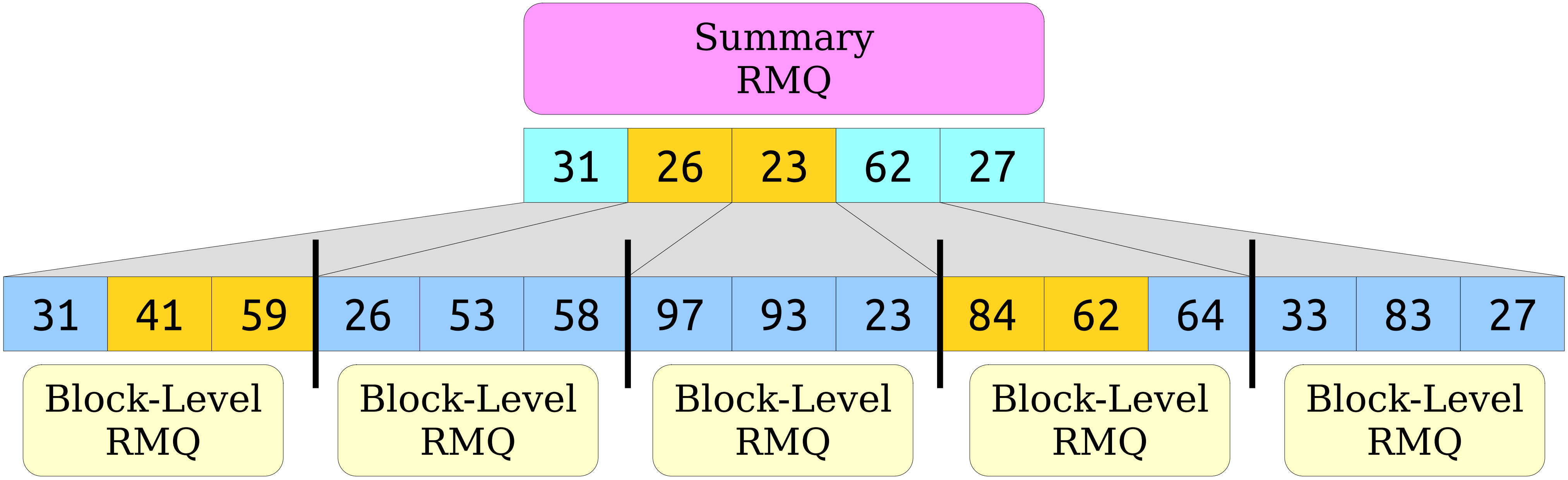
Blocking Revisited



This is just RMQ inside the blocks!

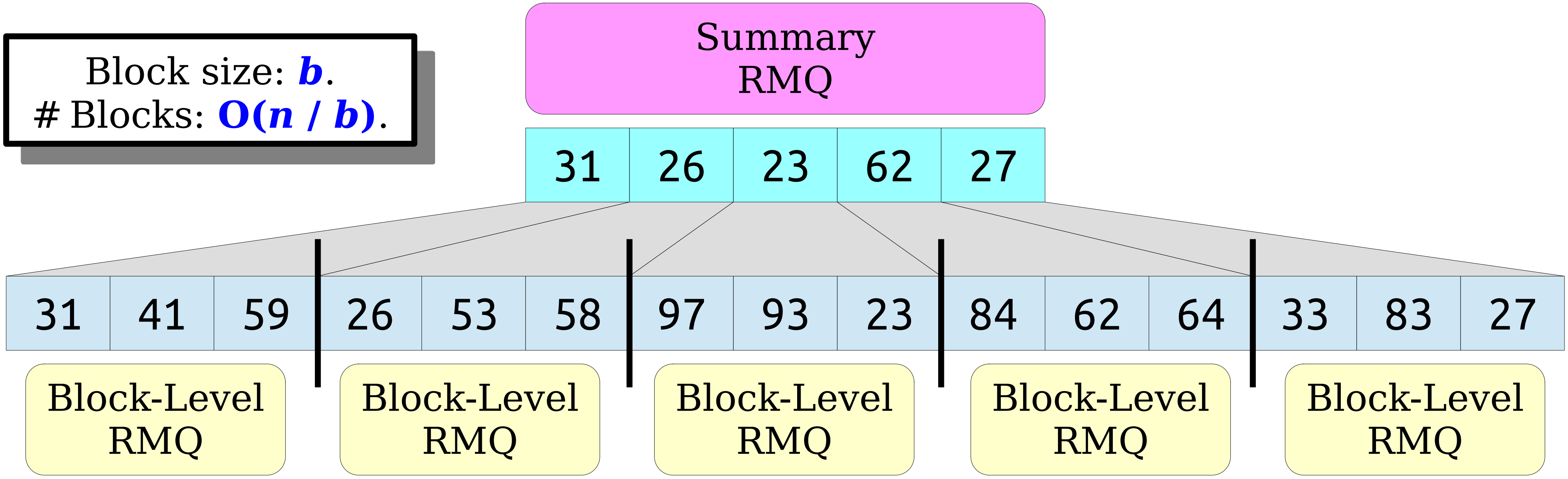
The Framework

- Split the input into blocks of size b.
- Form an array of the block minima.
- Construct a “summary” RMQ structure over the block minima.
- Construct “block” RMQ structures for each block.
- Aggregate the results together.



Analyzing Efficiency

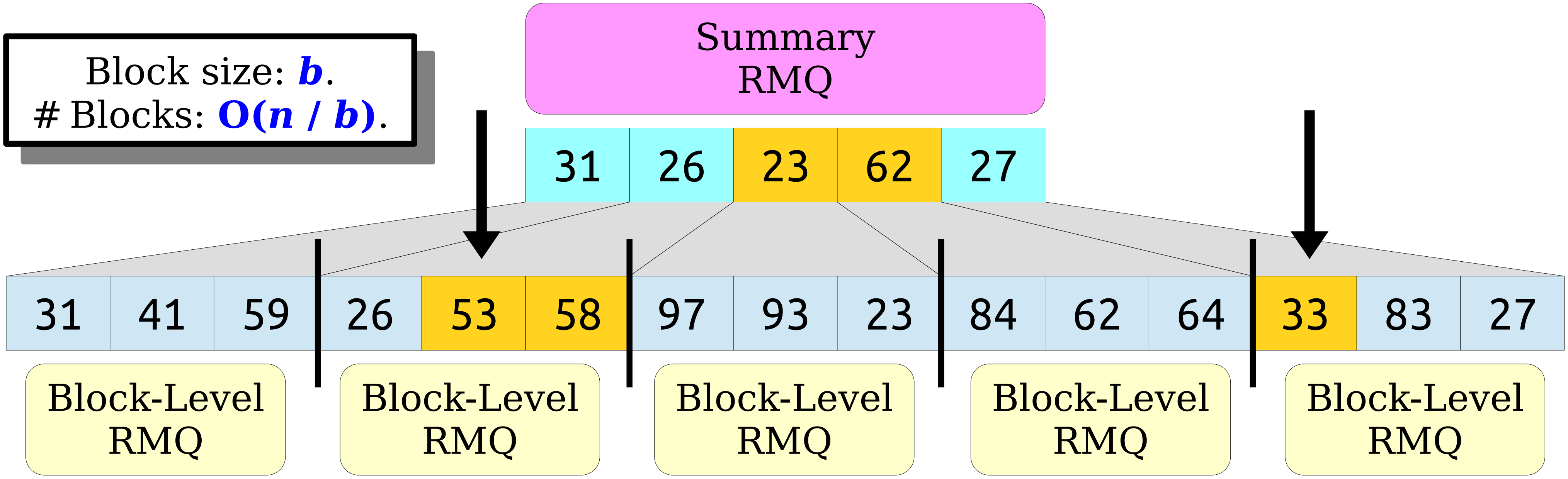
- Suppose we use a $\langle p_1(n), q_1(n) \rangle$-time RMQ for the summary RMQ and a $\langle p_2(n), q_2(n) \rangle$-time RMQ for each block, with block size b.
- What is the preprocessing time for this hybrid structure?
  - $\mathbf{O}(n)$ time to compute the minima of each block.
  - $\mathbf{O}(p_1(n / b))$ time to construct RMQ on the minima.
  - $\mathbf{O}((n / b)\ p_2(b))$ time to construct the block RMQs.
- Total construction time is $\mathbf{O}(n + p_1(n / b) + (n / b)\ p_2(b))$.



Analyzing Efficiency

- Suppose we use a ⟨$p_1(n)$, $q_1(n)$⟩-time RMQ for the summary RMQ and a ⟨$p_2(n)$, $q_2(n)$⟩-time RMQ for each block, with block size b.
- What is the query time for this hybrid structure?
  - $\mathbf{O(q_1(n / b))}$ time to query the summary RMQ.
  - $\mathbf{O(q_2(b))}$ time to query the block RMQs.
- Total query time: $\mathbf{O(q_1(n / b) + q_2(b))}$.

Block size: $\mathbf{b}$.
Blocks: $\mathbf{O(n / b)}$.



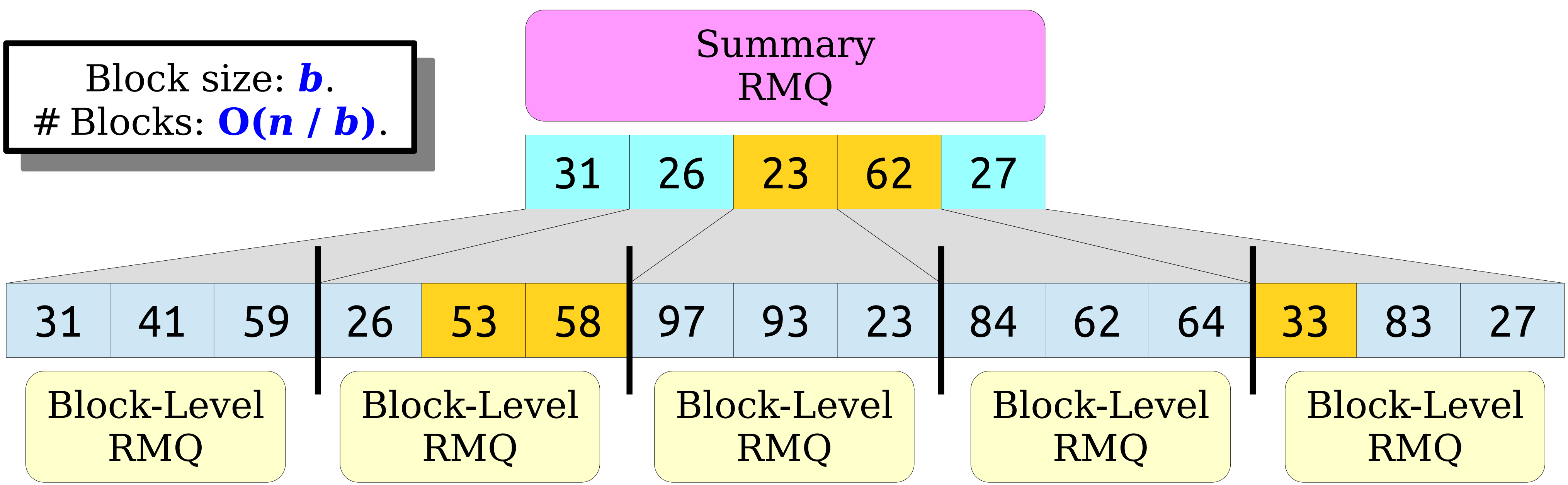
Analyzing Efficiency

- Suppose we use a $\langle p_1(n), q_1(n)\rangle$-time RMQ for the summary RMQ and a $\langle p_2(n), q_2(n)\rangle$-time RMQ for each block, with block size b.
- Hybrid preprocessing time:

$$\mathbf{O(n + p_1(n / b) + (n / b)p_2(b))}$$

- Hybrid query time:

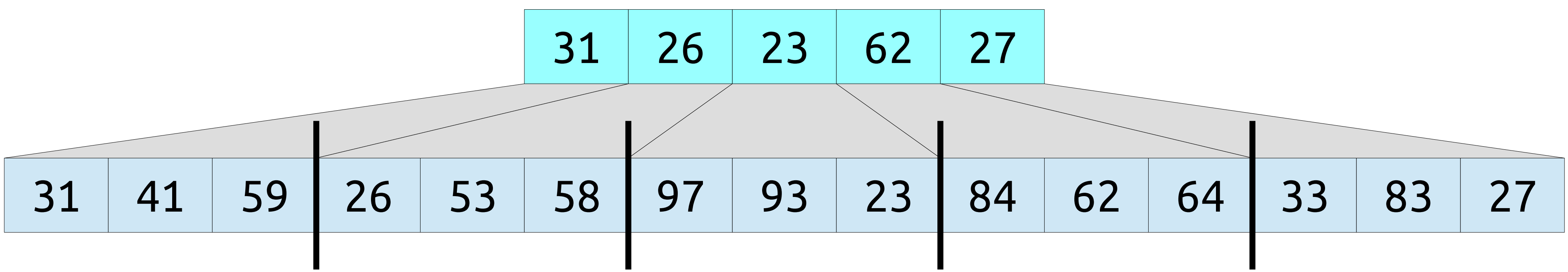
$$\mathbf{O(q_1(n / b) + q_2(b))}$$



A Sanity Check

- The ⟨O(n), O($n^{1/2}$)⟩ block-based structure from earlier uses this framework with the ⟨O(1), O(n)⟩ no-preprocessing RMQ structure and $b = n^{1/2}$.

Do no further preprocessing than just computing the block minima.



Don’t do anything fancy per block. Just do linear scans over each of them.

A Sanity Check

- The $\langle O(n), O(n^{1/2})\rangle$ block-based structure from earlier uses this framework with the $\langle O(1), O(n)\rangle$ no-preprocessing RMQ structure and $b = n^{1/2}$.
- According to our formulas, the preprocessing time should be

$$\begin{aligned} &O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ =\ &O(n + 1 + n / b) \\ =\ &\mathbf{O(n)} \end{aligned}$$

- The query time should be

$$\begin{aligned} &O(q_1(n / b) + q_2(b)) \\ =\ &O(n / b + b) \\ =\ &\mathbf{O(n^{1/2})} \end{aligned}$$

- Looks good so far!

For Reference

$p_1(n) = O(1)$
$q_1(n) = O(n)$

$p_2(n) = O(1)$
$q_2(n) = O(n)$

$b = n^{1/2}$

An Observation

- We can use any data structures we'd like for the summary and block RMQs.
- Suppose we use an $\langle O(n \log n), O(1) \rangle$ sparse table for the summary RMQ.
- If the block size is b, the time to construct a sparse table over the (n / b) blocks is $\mathbf{O((n / b) \log (n / b))}$.
- ***Cute trick:*** If $\mathbf{b = \Theta(\log n)}$, the time to construct a sparse table over the minima is

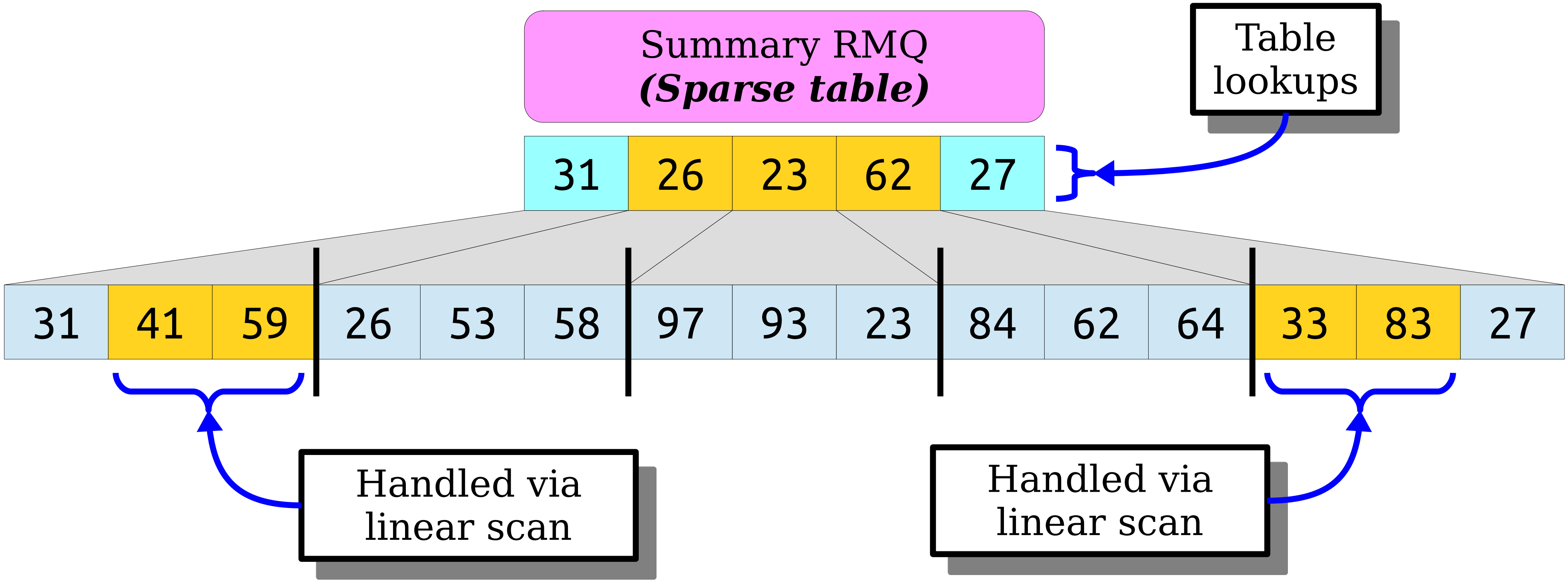
$$O((n / \log n) \log (n / \log n))$$

$$= O((n / \log n) \log n) \qquad \textit{(O is an upper bound)}$$

$$= \mathbf{O(n)}. \qquad \textit{(logs cancel out)}$$

One Possible Hybrid

- Set the block size to log *n*.
- Use a sparse table for the summary RMQ.
- Use the "no preprocessing" structure for each block.



One Possible Hybrid

- Set the block size to log n.
- Use a sparse table for the summary RMQ.
- Use the "no preprocessing" structure for each block.
- Preprocessing time:

$$
\begin{aligned}
& O(n + p_1(n / b) + (n / b)\, p_2(b)) \\
= {} & O(n + n + n / b) \\
= {} & \mathbf{O(\boldsymbol{n})}
\end{aligned}
$$

- Query time:

$$
\begin{aligned}
& O(q_1(n / b) + q_2(b)) \\
= {} & O(1 + b) \\
= {} & \mathbf{O(\log \boldsymbol{n})}
\end{aligned}
$$

- An **⟨O(*n*), O(log *n*)⟩** solution!

For Reference

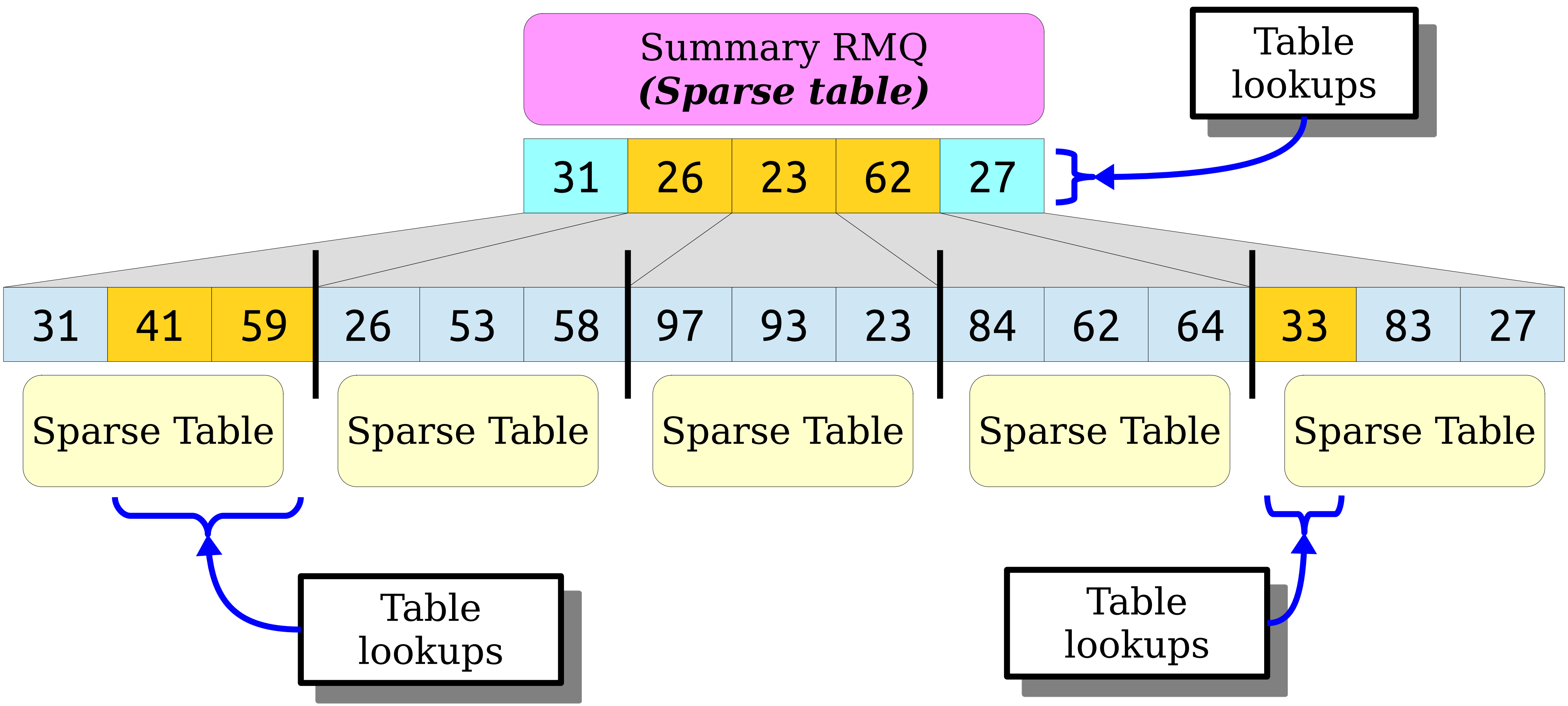
$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(1)$
$q_2(n) = O(n)$

$b = \log n$

Another Hybrid

- Let's suppose we use the ⟨O(n log n), O(1)⟩ sparse table for both the summary and block RMQ structures with a block size of log n.



Another Hybrid

- Let's suppose we use the $\langle O(n \log n), O(1)\rangle$ sparse table for both the summary and block RMQ structures with a block size of $\log n$.
- The preprocessing time is

$$\begin{aligned} & O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ =\ & O(n + n + (n / b)\, b \log b) \\ =\ & O(n + n \log b) \\ =\ & \mathbf{O(n \log \log n)} \end{aligned}$$

- The query time is

$$\begin{aligned} & O(q_1(n / b) + q_2(b)) \\ =\ & \mathbf{O(1)} \end{aligned}$$

- We have an **$\langle O(n \log \log n), O(1)\rangle$** solution to RMQ!

For Reference

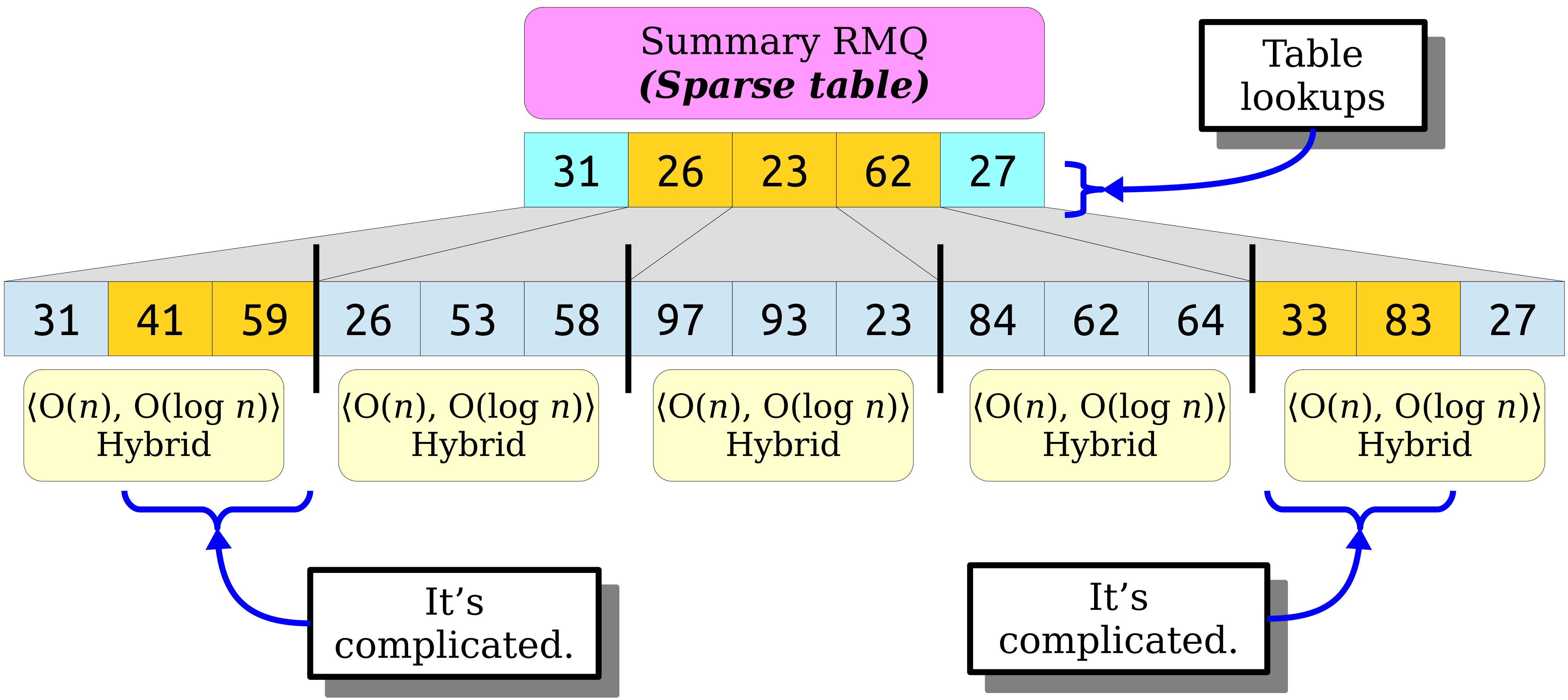
$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n \log n)$
$q_2(n) = O(1)$

$b = \log n$

One Last Hybrid

- Suppose we use a sparse table for the summary RMQ and the ⟨O(n), O(log n)⟩ solution for the block RMQs. Let's choose b = log n.



One Last Hybrid

- Suppose we use a sparse table for the summary RMQ and the ⟨O(n), O(log n)⟩ solution for the block RMQs. Let's choose $b = \log n$.
- The preprocessing time is

$$\begin{aligned} & O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ =\ & O(n + n + (n / b)\, b) \\ =\ & \mathbf{O(n)} \end{aligned}$$

- The query time is

$$\begin{aligned} & O(q_1(n / b) + q_2(b)) \\ =\ & O(1 + \log b) \\ =\ & \mathbf{O(\log \log n)} \end{aligned}$$

- We have an **⟨O(n), O(log log n)⟩** solution to RMQ!

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n)$
$q_2(n) = O(\log n)$

$b = \log n$

Where We Stand

- We've seen a bunch of RMQ structures today:
  - No preprocessing: $\langle O(1), O(n) \rangle$
  - Full preprocessing: $\langle O(n^2), O(1) \rangle$
  - Block partition: $\langle O(n), O(n^{1/2}) \rangle$
  - Sparse table: $\langle O(n \log n), O(1) \rangle$
  - Hybrid 1: $\langle O(n), O(\log n) \rangle$
  - Hybrid 2: $\langle O(n \log \log n), O(1) \rangle$
  - Hybrid 3: $\langle O(n), O(\log \log n) \rangle$

Where We Stand

We've seen a bunch of RMQ structures today:

- No preprocessing: $\langle O(1), O(n)\rangle$
- Full preprocessing: $\langle O(n^2), O(1)\rangle$
- Block partition: $\langle O(n), O(n^{1/2})\rangle$
- Sparse table: $\langle O(n \log n), O(1)\rangle$
- Hybrid 1: $\langle O(n), O(\log n)\rangle$
- Hybrid 2: $\langle O(n \log \log n), O(1)\rangle$
- Hybrid 3: $\langle O(n), O(\log \log n)\rangle$

Where We Stand

We've seen a bunch of RMQ structures today:

- No preprocessing: $\langle O(1), O(n) \rangle$
- Full preprocessing: $\langle O(n^2), O(1) \rangle$
- Block partition: $\langle O(n), O(n^{1/2}) \rangle$
- Sparse table: $\langle O(n \log n), O(1) \rangle$
- Hybrid 1: $\langle O(n), O(\log n) \rangle$
- Hybrid 2: $\langle O(n \log \log n), O(1) \rangle$
- Hybrid 3: $\langle O(n), O(\log \log n) \rangle$

Is there an $\langle O(n), O(1) \rangle$ solution to RMQ?

Yes!

Next Time

- ***Cartesian Trees***
  - A data structure closely related to RMQ.
- ***The Method of Four Russians***
  - A technique for shaving off log factors.
- ***The Fischer-Heun Structure***
  - A clever, asymptotically optimal RMQ structure.